HYPER VERSION

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Four dimensional $\mathcal{N} = 1$ supersymmetrization of $\mathcal{R}^4$ in superspace

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Abstract: We write an action, in four dimensional $\mathcal{N} = 1$ curved superspace, which contains a pure $\mathcal{R}^4$ term with a coupling constant. Starting from the off-shell solution of the Bianchi identities, we compute the on-shell torsions and curvatures with this term. We show that their complete solution includes, for some of them, an infinite series in the $\mathcal{R}^4$ coupling constant, which can only be computed iteratively. We explicitly compute the superspace torsions and curvatures up to second order in this coupling constant. Finally, we comment on the lifting of this result to higher dimensions.

Keywords: Supergravity Models, Supersymmetric Effective Theories, Superspaces.

Contents

1. Introduction

The supersymmetrization of higher-derivative terms has been object of research for a long time. Green and Schwarz proved that, in order to eliminate anomalies from type I supergravity coupled to super Yang-Mills theories, one needs the Green-Schwarz mechanism [1, 2], by which the supergravity two-form field strength is modified with the subtraction of the Lorentz Chern-Simons term. This mechanism violates local supersymmetry and originates an action with a $\mathcal{R}^2$ term.

In order to cancel the anomalies, it is also necessary to introduce bosonic higher-derivative counterterms (which include, for instance, $\mathcal{R}^4$ terms to cancel gravitational anomalies). These counterterms have been shown to originate naturally in string theory [2]. They should then be part of the string theory low energy effective action, but they have to be introduced by hand on the supergravity action. Obviously they also break supersymmetry.

Purely "stringy" $\mathcal{R}^4$ terms also show up in the low energy field theory effective action of both type II and heterotic string theories, as was shown in [3, 4] by computing four graviton scattering amplitudes and in [5] by calculating loop corrections to supersymmetric sigma models (the requirement that their β function vanishes determines the equations of motion of bosonic background fields, from which one determines the effective action).

The presence of those terms in M-theory has also been proven by one- and two-loop superparticle scattering calculations. The complete $\mathcal{R}^4$ term arises from four-graviton scattering in eleven-dimensional supergravity. The quantum contributions to this process were analyzed at one-loop in eleven-dimensional supergravity compactified on S^1 [6] and T^2 [6, 7]. The latest results were extended to two loops in [8].

In the three cases we have described, we have $\mathcal{R}^2$ and $\mathcal{R}^4$ terms which are not supersymmetric. The requirement of having a supersymmetric anomaly-free effective action motivates the supersymmetrization of these terms.

The supersymmetrization of the Lorentz Chern-Simons terms has already been made [9]–[12] (see also the review [13] for a more complete list of references); the corresponding result for the $\mathcal{R}^4$ terms in string and M-theory effective actions is still under study, though some results exist both in ten [10, 14] and eleven dimensions [15, 16]. In reference [14] there is a systematic discussion of the different types of $\mathcal{R}^4$ superinvariants one may have.

All these results and claims should also be valid in four dimensions, when one dimensionally reduces from ten or eleven. It is useful to consider the same problems in four-dimensional $\mathcal{N} = 1$ supergravity. What makes these problems easier, in principle, in this last case, is the knowledge of auxiliary fields, and the existence of a completely off-shell formulation of the theory in superspace. Also, the solution of these problems provides important information and consistency tests for the higher dimensional problems.

The coupling of the Lorentz Chern-Simons term to four-dimensional $\mathcal{N} = 1$ supergravity consistent with supersymmetry and dimensional reduction from string theory has already been worked out [17, 18, 19]. In these references, a linear multiplet is coupled to supergravity. The scalar belonging to that multiplet plays the role of the dilaton in the dimensionally reduced theory; the two-index gauge field in the same multiplet is also analogous to the ten-dimensional two-form whose field strength is corrected the Lorentz Chern-Simons term.

The problem of supersymmetrizing $\mathcal{R}^4$ in $\mathcal{N} = 1$ four-dimensional supergravity had never been worked out. From what has been said, this problem reduces then to simply writing the appropriate action in superspace and deriving the respective torsions and curvatures. That is what is done in this paper. In a future paper, we will present the complete action in x space.

In any case, already from the results of this paper we can conclude that it requires an infinite number of terms to achieve the complete supersymmetrization (after the elimination of auxiliary fields), i.e. to go “on-shell”. The same is valid for writing the superspace torsions and curvatures. Those terms are part of an infinite series in the $\mathcal{R}^4$ coupling constant, which cannot be written explicitly. In this paper, we compute the superspace torsions and curvatures up to second order in this coupling constant.

In section 2 we review briefly how to derive field equations in superspace for pure $\mathcal{N} = 1, d = 4$ supergravity.

In section 3 we motivate and write the superspace action containing the $\mathcal{R}^4$ term.

In section 4 we determine the field equations from this action. We show that we have two field equations, the solutions of which being respectively a polynomial and an infinite series.

Finally, in section 5 we compare this case with other higher-derivative superinvariants known in the literature, and we comment on the implications of our result for the determination of effective actions for superstring/M theory.

In appendix A we present our choice of conventions.

2. A review of pure $\mathcal{N} = 1, d = 4$ supergravity in superspace

In this section, we make a brief review of the superspace formulation of pure $\mathcal{N} = 1$, $d = 4$ supergravity, with emphasis on the process of deriving the field equations. None of the results mentioned in this section is original; we just include them here for the reader to be acquainted with them (written in our conventions), and because they are essencial for the rest of the paper, which is a generalization of these results to a superspace action including the $\mathcal{R}^4$ terms.

What is special about $\mathcal{N} = 1, d = 4$ supergravity is the existence of a completely off-shell formalism. This means that a complete set of auxiliary fields is known (actually, there exist three known choices). In superspace this means that, after imposing constraints on the torsions, we can completely solve the Bianchi identities without using the field equations [20]. Our choice of constraints on the torsions and the solutions to the Bianchi identities are listed, in our conventions, in appendix A.2. The main result is that we can express all the unconstrained torsions and curvatures as functions of three superfields R, $G_{A\dot{A}}$ and W_{ABC} (and their complex conjugates). These superfields have some properties and satisfy some differential constraints also listed in appendix A.2.

Another special feature of pure $\mathcal{N} = 1$ four-dimensional supergravity is that its action in superspace is known. It is written as the integral, over the whole superspace, of the superdeterminant of the supervielbein [21]:

$$I = \frac{1}{2\kappa^2} \int \int E d^4\theta d^4x, E = \mathrm{sdet} E_{\Lambda}{}^{M} . \qquad (2.1)$$

The following well known result is very useful when computing field equations in superspace from a given action:

$$\int E\nabla_M v^M \left(-\right)^M d^4x d^4\theta = 0\,. \tag{2.2}$$

It allows us to integrate by parts and discard the terms in the action which are full divergences. This result follows uniquely from the torsion constraints and the Bianchi identities.

Some of the constraints on the torsions allow us to express all spin connections in terms of supervielbeins. The other torsion constraints represent constraints on the supervielbein. One can express ${E_m}^\Lambda$ in terms of ${E_A}^\Lambda$ and ${E_{\dot{A}}}^\Lambda$, which can be expressed in terms of prepotentials [22]. But these constraints do not depend on any action, and should be preserved when we vary the supervielbeins and superconnections, in order to derive the field equations for any action we take. Therefore we define, according to the original work [21] and the review [23],

$$\begin{aligned} {H_M}^N &= {E_M}^\Lambda \delta {E_\Lambda}^N \Rightarrow \delta {E_\Lambda}^N = {E_\Lambda}^M {H_M}^N & (2.3)\\ {\Phi_{MN}}^P &= {E_M}^\Lambda \delta {\Omega_{\Lambda N}}^P & (2.4)\end{aligned}$$

as arbitrary variations of the supervielbein and the superconnection, respectively, subject to the condition that the torsion constraints should remain satisfied. ${\Phi_{MN}}^P$ is Lorentz-valued, like ${\Omega_{MN}}^P$: ${\Phi_{MB}}^C = -\frac{1}{4}\Phi_{Mmn}\left(\sigma^{mn}\right)_B{}^C = \frac{1}{4}{\Phi_{MB\dot{B}}}^{C\dot{B}}$.

From the constrained variations (2.3), (2.4) and the definition (A.10) of torsion, we derive the constrained variation

$$\begin{aligned} \delta {T_{MN}}^R = &-{H_M}^S {T_{SN}}^R + \left(-\right)^{MN} {H_N}^S {T_{SM}}^R + {T_{MN}}^S {H_S}^R - \\ &-\nabla_M {H_N}^R + \left(-\right)^{MN} \nabla_N {H_M}^R + {\Phi_{MN}}^R - \left(-\right)^{MN} {\Phi_{NM}}^R\,. \end{aligned} \tag{2.5}$$

The equations for $\delta {T_{MN}}^R$ are invariant under the following two gauge transformations:

$$\begin{aligned} \delta {H_M}^N &= \nabla_M \xi^N - \xi^P {T_{PM}}^N \\ \delta {\Phi_{MN}}^P &= \xi^Q {R_{QMN}}^P \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \delta {H_M}^N &= {X_M}^N \\ \delta {\Phi_{MN}}^P &= \nabla_M {X_N}^P\,. \end{aligned} \tag{2.7}$$

The method for deriving, from a given action, its field equations in superspace has been exposed in [21, 23, 24, 25]. For completeness, we summarize it here. The main idea is to determine the whole set of ${H_M}^N$ and ${\Phi_{MN}}^P$. This can be achieved by applying (2.5) to the constrained torsions. From what we mentioned before, their variation should be zero, which implies a set of algebraic equations from which $H_M{}^N$ and $\Phi_{MN}{}^P$ can be determined in terms of the unconstrained supertorsions and supercurvatures (which means in terms of R, $G_{A\dot A}$ and W_{ABC} and their complex conjugates), and some extra arbitrary superfields. Since these extra superfields are arbitrary, δI can only vanish if their coefficients vanish. The vanishing of these coefficients is equivalent to the superspace field equations.

We will briefly review the pure supergravity case, before going to the $\mathcal{R}^4$ action. Details and derivations can be found in [21, 23, 24, 25]. The variation of the action (2.1) is simply given by

$$2\kappa^2\delta I = \int E\left(\frac{1}{2}H_{A\dot A}{}^{A\dot A} - H_A{}^{A} - H_{\dot A}{}^{\dot A}\right) d^4x d^4\theta\,. \tag{2.8}$$

From $\delta T_{AB}{}^{\dot C} = 0$, $\delta T_{AB}{}^{m} = 0$, and by fixing the gauge invariance (2.6), it has been shown that we may write

$$H_{\dot B k} = -\nabla_{\dot B}\chi_k,\; H_{Bk} = \nabla_B\chi_k \tag{2.9}$$

$$H_{\dot B A} = \frac{i}{24}\overline{R}\chi_{A\dot B},\; H_{A\dot B} = -\frac{i}{24}R\chi_{A\dot B} \tag{2.10}$$

χ_k is an arbitrary, *pure imaginary* superfield.

It has also been shown that we have the following parameterization:[1]

$$H_k{}^k - H_A{}^A + \nabla^k\chi_k - 2i\chi^k G_k = \left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\overline{U} \tag{2.11}$$

$$H_k{}^k - H_{\dot A}{}^{\dot A} - \nabla^k\chi_k - 2i\chi^k G_k = \left(\nabla^2 + \frac{1}{3}R\right)U \tag{2.12}$$

$\left(\nabla^2 + \frac{1}{3}R\right)$ is, in our notations, the chiral projector for a scalar superfield. U is an arbitrary superfield, and $\overline{U}$ its complex conjugate.

The final result for (2.8) is [23][2]

$$\begin{aligned} H_k{}^k - H_A{}^A - H_{\dot A}{}^{\dot A} = {} & \frac{i}{12}\left(\nabla^{\dot A}\nabla^{A} - \nabla^{A}\nabla^{\dot A}\right)\chi_{A\dot A} + \frac{2}{3}i\chi^{A\dot A}G_{A\dot A} + \\ & + \frac{1}{3}\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\overline{U} + \frac{1}{3}\left(\nabla^2 + \frac{1}{3}R\right)U\,. \end{aligned} \tag{2.13}$$

Replacing in (2.8), and recalling (2.2),

$$2\kappa^2\delta I = \int E\left(\frac{2}{3}i\chi^{A\dot A}G_{A\dot A} + \frac{1}{9}\overline{RU} + \frac{1}{9}RU\right) d^4x d^4\theta = 0\,. \tag{2.14}$$

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[1] Equations (2.9), (2.10), (2.11) and (2.12) are derived in [24, 25] and were told to us by P. van Nieuwenhuizen.

[2] We will present our derivation of (2.13) in section 4, after we have computed the remaining components of $H_M{}^N$.

Since $U, \chi^{A\dot{A}}$ are completely arbitrary, there's no other possibility than to have

$$G_{A\dot{A}} = 0 \tag{2.15}$$
$$R = 0\,. \tag{2.16}$$

These are the field equations for pure $\mathcal{N} = 1$ four-dimensional supergravity.

3. The supersymmetric $\mathcal{R}^4$ action

To the action (2.1), we are adding (supersymmetric) $\mathcal{R}^4$ correction terms. Because of the field equations (2.15), (2.16) it does not make sense to add terms to (2.1) which are proportional to R or $G_{A\dot{A}}$ (the field equations would only get perturbative corrections which would not affect the unperturbed solutions (2.15), (2.16)). Therefore, the terms we are looking for must contain W_{ABC} and $W_{\dot{A}\dot{B}\dot{C}}$. Indeed, it is well known [26] that the component expansion of W_{ABC} includes the term[3]

$$\nabla_{\underline{A}} W_{\underline{BCD}}\big| = \frac{1}{8}\mathcal{R}_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}} + \cdots = -\frac{1}{2}\mathcal{W}_{ABCD} + \cdots \tag{3.1}$$

$\mathcal{W}_{ABCD}$ is the Weyl tensor in spinor notation, a Lorentz-irreducible component of the Riemannn tensor [27, chapters 4 and 13].

One can easily (using the solutions to the Bianchi identities and (A.24), (A.25)) derive the following formula:

$$\begin{aligned}\nabla^2 W^2 = & -2\left(\nabla^A W^{BCD}\right)\nabla_{\underline{A}} W_{\underline{BCD}} + 12\left(\nabla^m G^n\right)\nabla_m G_n - \\ & -12\left(\nabla^m G^n\right)\nabla_n G_m - 12i\varepsilon^{mnrs}\left(\nabla_m G_n\right)\nabla_r G_s - \\ & -\frac{5}{3}RW^2 - 20W^{ABC}G_A{}^{\dot{A}}\nabla_B G_{C\dot{A}} + 8iW^{ABC}\nabla_A{}^{\dot{A}}\nabla_B G_{C\dot{A}}\,. \end{aligned} \tag{3.2}$$

We see that the θ^2 component of W^2 has a $\mathcal{W}^2$ term. This way, a supersymmetric action which includes $\mathcal{R}^4$ (actually, $\mathcal{W}^4$) should have (because of hermiticity) a $W^2\overline{W}^2$ term. The action we are then considering[4] is written, in superspace, in the following way:

$$I = \frac{1}{2\kappa^2}\int\int E\left(1 + \alpha W^2\overline{W}^2\right)d^4\theta d^4x \tag{3.3}$$

α is a coupling constant, a perturbative parameter of mass dimension -6. From now on, in each formula we write, the limit $\alpha = 0$ is the limit of pure supergravity.

It is important to know the exact expansion in components of (3.3). We leave the details for another publication [28]. Anyway, it is useful to mention here some

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[3] The underlined indices are symmetrized with weigth one; the vertical bar means the $\theta = 0$ component.

[4] In a future publication [28] we will discuss other possibilities for supersymmetric $\mathcal{R}^4$ actions.

aspects. It is well known that, in the old minimal formulation of supergravity [29, 30], there are no fermionic auxiliary fields. The auxiliary fields are then a scalar M, a pseudoscalar N and a vector A_m, which are zero on-shell (in pure supergravity). The following identification is valid [31]:

$$\overline{R}\big| = 4\,(M + iN) \tag{3.4}$$

$$G_{A\dot{A}}\big| = \frac{1}{3} A_{A\dot{A}}\,. \tag{3.5}$$

It agrees with the field equations (2.15), (2.16) of pure supergravity. But if we consider our higher-derivative action (3.2), we see that the θ^2 component of W^2 has derivatives of $G_{A\dot{A}}$. If we expand then (3.3) in components, we will also get a higher-derivative action in the auxiliary field A_m, which means a complicated nonalgebraic field equation for it.

It is worth mentioning that the piece of (3.3) proportional to α, i.e. the $\mathcal{R}^4$ correction to pure supergravity, is also the subleading correction to the Weyl supergravity action in a recently proposed Born-Infeld-Weyl supergravity action [32].

4. The superspace field equations, torsions and curvatures

Now we are ready to follow the same procedure as in section 2, but this time with respect to the action (3.3). We will need the whole set of components of ${H_M}^N$ and all but one of ${\Phi_{MN}}^P$. We start by this computation.

4.1 The full computation of ${H_M}^N$ and ${\Phi_{MN}}^P$

From the variation of the constrained torsions, we get

$$\begin{aligned}
\delta {T_{AB}}^C = 0 \Rightarrow\ & -\frac{1}{2} {H_{\underline{A}}}^{D\dot{D}} {T_{D\dot{D}\ \underline{B}}}^C - \nabla_{\underline{A}} {H_{\underline{B}}}^C + {\Phi_{\underline{AB}}}^C = 0 \\
\delta {T_{A\dot{B}}}^{\dot{C}} = 0 \Rightarrow\ & -\frac{1}{2} {H_A}^{D\dot{D}} {T_{D\dot{D}\dot{B}}}^{\dot{C}} - \frac{1}{2} {H_{\dot{B}}}^{D\dot{D}} {T_{D\dot{D}A}}^{\dot{C}} + {T_{A\dot{B}}}^n {H_n}^{\dot{C}} - \\
& -\nabla_A H_{\dot{B}}^{\dot{C}} - \nabla_{\dot{B}} {H_A}^{\dot{C}} + {\Phi_{A\dot{B}}}^{\dot{C}} = 0 \\
\delta {T_{A\dot{B}}}^m = 0 \Rightarrow\ & -{H_A}^C {T_{C\dot{B}}}^m - {H_{\dot{B}}}^{\dot{C}} {T_{\dot{C}A}}^m + T_{A\dot{B}}^{\ n} {H_n}^m - \\
& -\nabla_A H_{\dot{B}}^m - \nabla_{\dot{B}} H_A^m = 0 \\
\delta {T_{Am}}^n = 0 \Rightarrow\ & {H_m}^{\dot{B}} {T_{\dot{B}A}}^n + {T_{Am}}^B H_B^n + {T_{Am}}^{\dot{B}} H_{\dot{B}}^n - \nabla_A {H_m}^n + \nabla_m H_A^n + \\
& +\Phi_{Am}^{\ n} = 0 \\
\delta {T_{mn}}^p = 0 \Rightarrow\ & {T_{mn}}^A H_A^p + {T_{mn}}^{\dot{A}} H_{\dot{A}}^p - \nabla_m {H_n}^p + \nabla_n {H_m}^p + \\
& +{\Phi_{mn}}^p - {\Phi_{nm}}^p = 0\,.
\end{aligned} \tag{4.1}$$

From $\delta {T_{Am}}^n = 0$, we get

$$\begin{aligned}
{\Phi_{AB\dot{B}}}^{C\dot{C}} = & -{H_{B\dot{B}}}^{\dot{A}} {T_{\dot{A}A}}^{C\dot{C}} - {T_{AB\dot{B}}}^D H_D^{C\dot{C}} - {T_{AB\dot{B}}}^{\dot{D}} {H_{\dot{D}}}^{C\dot{C}} + \\
& +\nabla_A H_{B\dot{B}}^{C\dot{C}} - \nabla_{B\dot{B}} H_A^{C\dot{C}}\,.
\end{aligned} \tag{4.2}$$

From $\delta T_{mn}{}^p = 0$ we get an algebraic equation for $\Phi_{mn}{}^p$. This specific computation is not necessary for the variation of our action (see section 4.2); once we determine $H_m{}^p$, as we will, and knowing $H_A{}^p$, this computation becomes straightforward.

From $\delta T_{A\dot B}{}^{\dot C} = 0$ we get

$$\begin{aligned}\Phi_{A\dot B}{}^{\dot C} = \frac{1}{4}\Phi_{AB\dot B}{}^{B\dot C} = {}& \frac{1}{2}H_A{}^{D\dot D}T_{D\dot D\dot B}{}^{\dot C} + \frac{1}{2}H_{\dot B}{}^{D\dot D}T_{D\dot D A}{}^{\dot C} + \\ & + 2iH_{A\dot B}{}^{\dot C} + \nabla_A H_{\dot B}{}^{\dot C} + \nabla_{\dot B} H_A{}^{\dot C}\end{aligned} \tag{4.3}$$

From $\delta T_{A\dot B}{}^m = 0$

$$2iH_A{}^C\sigma_{C\dot B}{}^m + 2iH_{\dot B}{}^{\dot C}\sigma_{A\dot C}{}^m - 2iH_{A\dot B}{}^m - \nabla_A H_{\dot B}^m - \nabla_{\dot B}H_A^m = 0 \tag{4.4}$$

from which we get the equation

$$H_{A\dot A}{}^{B\dot B} = 2\left(H_A{}^B\varepsilon_{\dot A}{}^{\dot B} + H_{\dot A}{}^{\dot B}\varepsilon_A{}^B\right) + \frac{i}{2}\left(\nabla_A H_{\dot A}{}^{B\dot B} + \nabla_{\dot A}H_A{}^{B\dot B}\right). \tag{4.5}$$

We can use the gauge invariance (2.7) to gauge away the symmetric part of H_{AB}:

$$\begin{aligned}H_A{}^B &= \frac{1}{2}H_C{}^C\delta_A{}^B \\ H_{\dot A}{}^{\dot B} &= \frac{1}{2}H_{\dot C}{}^{\dot C}\delta_{\dot A}{}^{\dot B}.\end{aligned} \tag{4.6}$$

Replacing (2.9) and (4.6) into (4.5), we get

$$H_{A\dot A}{}^{B\dot B} = \left(H_C{}^C + H_{\dot C}{}^{\dot C}\right)\delta_A{}^B\delta_{\dot A}{}^{\dot B} + \frac{i}{2}\left(\nabla_A\nabla_{\dot A} - \nabla_{\dot A}\nabla_A\right)\chi^{B\dot B}. \tag{4.7}$$

Tracing the last equation for $H_{A\dot A}{}^{B\dot B}$ we obtain

$$H_k{}^k = 2\left(H_A{}^A + H_{\dot A}{}^{\dot A}\right) - \frac{i}{4}\left(\nabla_A\nabla_{\dot A} - \nabla_{\dot A}\nabla_A\right)\chi^{A\dot A}. \tag{4.8}$$

Combining this last equation with the previous equations (2.11) and (2.12), we can solve for

$$\begin{aligned}H_k{}^k &= \frac{8}{3}i\chi^k G_k + \frac{2}{3}\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\overline{U} + \frac{2}{3}\left(\nabla^2 + \frac{1}{3}R\right)U + \\ &\quad + \frac{i}{12}\left(\nabla_A\nabla_{\dot A} - \nabla_{\dot A}\nabla_A\right)\chi^{A\dot A} \\ H_A{}^A &= \frac{2}{3}i\chi^k G_k - \frac{1}{3}\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\overline{U} + \frac{2}{3}\left(\nabla^2 + \frac{1}{3}R\right)U + \\ &\quad + \frac{i}{12}\left(\nabla_A\nabla_{\dot A} - \nabla_{\dot A}\nabla_A\right)\chi^{A\dot A} + \nabla^k\chi_k \\ H_{\dot A}{}^{\dot A} &= \frac{2}{3}i\chi^k G_k + \frac{2}{3}\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\overline{U} - \frac{1}{3}\left(\nabla^2 + \frac{1}{3}R\right)U + \\ &\quad + \frac{i}{12}\left(\nabla_A\nabla_{\dot A} - \nabla_{\dot A}\nabla_A\right)\chi^{A\dot A} - \nabla^k\chi_k.\end{aligned} \tag{4.9}$$

From these three expressions, (2.13) follows. Also, from (4.7), it follows that

$$\begin{aligned} H_{A\dot{A}}{}^{B\dot{B}} = & \left[\frac{1}{3}\left(\nabla^2 + \frac{1}{3}R\right)U + \frac{1}{3}\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\overline{U} + \frac{2}{3}i\chi^{C\dot{C}}G_{C\dot{C}} + \right. \\ & \left. + \frac{i}{6}\left(\nabla_C\nabla_{\dot{C}} - \nabla_{\dot{C}}\nabla_C\right)\chi^{C\dot{C}}\right]\varepsilon_A{}^B\varepsilon_{\dot{A}}{}^{\dot{B}} + \\ & + \frac{i}{2}\left(\nabla_{\dot{A}}\nabla_A - \nabla_A\nabla_{\dot{A}}\right)\chi^{B\dot{B}}\,. \end{aligned} \tag{4.10}$$

Replacing these expressions in (4.2) — actually its complex conjugate-, we can now solve for $H_{A\dot{A}}{}^B$:

$$\begin{aligned} H_{A\dot{A}}{}^B = & \frac{1}{6}\varepsilon_A{}^B\nabla_{\dot{A}}\left(\chi^{D\dot{D}}G_{D\dot{D}}\right) + \frac{i}{6}\left[\nabla_{\dot{A}}\left(\nabla^2 + \frac{1}{3}R\right)U\right]\varepsilon_A{}^B + \\ & + \frac{i}{4}\varepsilon_A{}^B\nabla_{\dot{A}}\left(\nabla^{D\dot{D}}\chi_{D\dot{D}}\right) + \frac{1}{24}\varepsilon_A{}^B\nabla_{\dot{A}}\left(\nabla_D\nabla_{\dot{D}} - \nabla_{\dot{D}}\nabla_D\right)\chi^{D\dot{D}} + \\ & + \frac{1}{8}\nabla_{\dot{A}}\left(\nabla_{\dot{D}}\nabla_A - \nabla_A\nabla_{\dot{D}}\right)\chi^{B\dot{D}} - \frac{1}{16}\overline{R}\nabla_A\chi^B{}_{\dot{A}} - \frac{i}{4}\nabla_{A\dot{B}}\nabla_{\dot{A}}\chi^{B\dot{B}} - \\ & - \frac{1}{24}\chi^B{}_{\dot{A}}\nabla_A\overline{R} + \frac{3}{16}G_A{}^{\dot{B}}\nabla_{\dot{B}}\chi^B{}_{\dot{A}} + \frac{1}{16}G_{A\dot{A}}\nabla_{\dot{B}}\chi^{B\dot{B}} - \\ & - \frac{1}{16}G_{A\dot{B}}\nabla_{\dot{A}}\chi^{B\dot{B}} - \frac{3}{8}G^{B\dot{B}}\nabla_{\dot{A}}\chi_{A\dot{B}} - \frac{3}{8}\varepsilon_A{}^B\left(\nabla_{\dot{A}}\chi^{C\dot{B}}\right)G_{C\dot{B}}\,. \end{aligned} \tag{4.11}$$

This finishes our computation of $H_M{}^N$. We see that we did not need to introduce any other arbitrary superfield for this computation: the superfields introduced in section 2 are enough, as expected, because we have three independent superfields and we want to find out relations between them. These relations should be generalizations of (2.15) and (2.16): $G_{A\dot{A}}$ and R should be functions of the independent superfield W_{ABC}, which is responsible to the correction terms in (3.3). Therefore we only have two equations and two arbitrary superfields.

4.2 The field equations in superspace

The variation of our action (3.3) is given by the superspace integral of

$$\begin{aligned} \delta\left[E\left(1 + \alpha W^2\overline{W}^2\right)\right] = & 2\alpha E\left(\overline{W}^2W^{ABC}\delta W_{ABC} + W^2W^{\dot{A}\dot{B}\dot{C}}\delta W_{\dot{A}\dot{B}\dot{C}}\right) + \\ & + \left(1 + \alpha W^2\overline{W}^2\right)\delta E\,. \end{aligned} \tag{4.12}$$

For this computation we obviously need the constrained variation of W_{ABC}. The details of this calculation are presented in appendix B, where we derive an equation for $\overline{W}^2W^{ABC}\delta W_{ABC}$ + h.c.. Having this result (which is eq. (B.9)), we should now integrate it in superspace (multiplied by E). In order to "factorize" $i\chi^{A\dot{A}}, U, \overline{U}$ to get the field equations, we should integrate by parts the terms which contain derivatives of these superfields. Following this procedure, we write

$$
\begin{aligned}
&\int E\Big[2\overline{W}^2 W^{ABC}\delta W_{ABC} + 2W^2 W^{\dot{A}\dot{B}\dot{C}}\delta W_{\dot{A}\dot{B}\dot{C}}\Big]\, d^4x d^4\theta = \\
&= \int E\bigg[i\chi^{A\dot{A}}\bigg[-2G_{A\dot{A}}W^2\overline{W}^2 + \frac{i}{8}\nabla^B\Big(\overline{R}\nabla^C{}_{\dot{A}}\Big(W_{ABC}\overline{W}^2\Big)\Big) + \\
&\qquad + \frac{1}{2}\left(\nabla_{\dot{D}}\nabla^B - \nabla^B\nabla_{\dot{D}}\right)\Big(\Big(\nabla_{\dot{A}}G^{C\dot{D}}\Big)W_{ABC}\overline{W}^2\Big) - \\
&\qquad - \frac{i}{4}\left(\nabla_{\dot{A}}\nabla^B - \nabla^B\nabla_{\dot{A}}\right)\nabla^{\dot{D}}\nabla^C{}_{\dot{D}}\Big(W_{ABC}\overline{W}^2\Big) + \\
&\qquad + \frac{1}{2}\nabla^{\dot{D}}\nabla^B{}_{\dot{A}}\nabla^C{}_{\dot{D}}\Big(W_{ABC}\overline{W}^2\Big) - \\
&\qquad - \frac{i}{12}\left(\nabla^B\overline{R}\right)\nabla^C{}_{\dot{A}}\Big(W_{ABC}\overline{W}^2\Big) + \\
&\qquad + \frac{3}{8}i\nabla^{\dot{D}}\Big(G^B{}_{\dot{D}}\nabla^C{}_{\dot{A}}\Big(W_{ABC}\overline{W}^2\Big)\Big) - \\
&\qquad - \frac{i}{8}\nabla_{\dot{A}}\Big(G^{B\dot{D}}\nabla^C{}_{\dot{D}}\Big(W_{ABC}\overline{W}^2\Big)\Big) - \\
&\qquad - \frac{5}{8}i\nabla^{\dot{D}}\Big(G^B{}_{\dot{A}}\nabla^C{}_{\dot{D}}\Big(W_{ABC}\overline{W}^2\Big)\Big) + \\
&\qquad + \frac{1}{12}\left(\nabla^B G^C{}_{\dot{A}}\right)\overline{R}W_{ABC}\overline{W}^2 - \\
&\qquad - \frac{3}{16}\nabla^B\Big(G^C{}_{\dot{A}}\overline{R}W_{ABC}\overline{W}^2\Big) + \frac{1}{8}\left(\nabla^B\overline{R}\right)G^C{}_{\dot{A}}W_{ABC}\overline{W}^2 + \\
&\qquad + \frac{3}{8}\left(\nabla_{\dot{A}}\nabla^B - \nabla^B\nabla_{\dot{A}}\right)\nabla^{\dot{D}}\Big(G^C{}_{\dot{D}}W_{ABC}\overline{W}^2\Big) + \\
&\qquad + \frac{3}{4}i\nabla^{\dot{D}}\nabla^C{}_{\dot{A}}\Big(G^B{}_{\dot{D}}W_{ABC}\overline{W}^2\Big) + \\
&\qquad + \frac{3}{8}\nabla^{\dot{D}}\Big(G^B{}_{\dot{A}}G^C{}_{\dot{D}}W_{ABC}\overline{W}^2\Big) + \text{h.c.}\bigg] - \\
&\quad - U\nabla^2\Big(W^2\overline{W}^2\Big) - \frac{U}{3}RW^2\overline{W}^2 - \overline{U}\overline{\nabla}^2\Big(W^2\overline{W}^2\Big) - \frac{\overline{U}}{3}\overline{R}W^2\overline{W}^2 + \\
&\quad + \text{full divergences}\bigg]\, d^4x d^4\theta\,. \qquad (4.13)
\end{aligned}
$$

Using the previous results (2.8), (2.13) and (4.13), and remembering that we can use (2.2) to discard the terms which are full divergences, we finally get for (4.12)

$$
\begin{aligned}
&\int \delta\Big[E\Big(1+\alpha W^2\overline{W}^2\Big)\Big]\, d^4x d^4\theta = \\
&= \alpha\int E\bigg\{\bigg[\frac{2}{3\alpha}G_{A\dot{A}} - \frac{4}{3}G_{A\dot{A}}W^2\overline{W}^2 - \\
&\qquad - \frac{1}{12}\left(\nabla_A\nabla_{\dot{A}} - \nabla_{\dot{A}}\nabla_A\right)\Big(W^2\overline{W}^2\Big) + \\
&\qquad + \bigg[\frac{1}{2}\left(\nabla_{\dot{D}}\nabla^B - \nabla^B\nabla_{\dot{D}}\right)\Big(\Big(\nabla_{\dot{A}}G^{C\dot{D}}\Big)W_{ABC}\overline{W}^2\Big) +
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{i}{8}\nabla^B\left(\overline{R}\nabla^C{}_{\dot A}\left(W_{ABC}\overline{W}^2\right)\right)-\\
&-\frac{i}{4}\left(\nabla_{\dot A}\nabla^B-\nabla^B\nabla_{\dot A}\right)\nabla^{\dot D}\nabla^C{}_{\dot D}\left(W_{ABC}\overline{W}^2\right)+\\
&+\frac{1}{2}\nabla^{\dot D}\nabla^B{}_{\dot A}\nabla^C{}_{\dot D}\left(W_{ABC}\overline{W}^2\right)-\\
&-\frac{i}{12}\left(\nabla^B\overline{R}\right)\nabla^C{}_{\dot A}\left(W_{ABC}\overline{W}^2\right)+\\
&+\frac{3}{8}i\nabla^{\dot D}\left(G^B{}_{\dot D}\nabla^C{}_{\dot A}\left(W_{ABC}\overline{W}^2\right)\right)-\\
&-\frac{i}{8}\nabla_{\dot A}\left(G^{B\dot D}\nabla^C{}_{\dot D}\left(W_{ABC}\overline{W}^2\right)\right)-\\
&-\frac{5}{8}i\nabla^{\dot D}\left(G^B{}_{\dot A}\nabla^C{}_{\dot D}\left(W_{ABC}\overline{W}^2\right)\right)-\\
&-\frac{3}{16}\nabla^B\left(G^C{}_{\dot A}\overline{R}W_{ABC}\overline{W}^2\right)+\\
&+\frac{1}{12}\left(\nabla^BG^C{}_{\dot A}\right)\overline{R}W_{ABC}\overline{W}^2+\\
&+\frac{1}{8}\left(\nabla^B\overline{R}\right)G^C{}_{\dot A}W_{ABC}\overline{W}^2+\\
&+\frac{3}{8}\left(\nabla_{\dot A}\nabla^B-\nabla^B\nabla_{\dot A}\right)\nabla^{\dot D}\left(G^C{}_{\dot D}W_{ABC}\overline{W}^2\right)+\\
&+\frac{3}{4}i\nabla^{\dot D}\nabla^C{}_{\dot A}\left(G^B{}_{\dot D}W_{ABC}\overline{W}^2\right)+\\
&+\frac{3}{8}\nabla^{\dot D}\left(G^B{}_{\dot A}G^C{}_{\dot D}W_{ABC}\overline{W}^2\right)+\text{h.c.}\Big]\Big]i\chi^{A\dot A}-\\
&-\frac{2}{3}U\overline{W}^2\nabla^2W^2-\frac{2}{9}URW^2\overline{W}^2+\frac{1}{9}\frac{UR}{\alpha}-\\
&-\frac{2}{3}\overline{U}W^2\overline{\nabla}^2\overline{W}^2-\frac{2}{9}\overline{UR}W^2\overline{W}^2+\frac{1}{9}\frac{\overline{UR}}{\alpha}\Big\}d^4xd^4\theta=\\
&=0\,.
\end{aligned}
\tag{4.14}
$$

The $U,\overline{U}$ field equations are immediately read:

$$
\begin{aligned}
\frac{2}{3}U\overline{W}^2\nabla^2W^2+\frac{2}{9}URW^2\overline{W}^2-\frac{1}{9}\frac{UR}{\alpha}&=0\\
\frac{R}{3}\left(2W^2\overline{W}^2-\frac{1}{\alpha}\right)&=-2\overline{W}^2\nabla^2W^2\,.
\end{aligned}
\tag{4.15}
$$

Since $W^5=0$, we get then the following (exact) result:

$$
\begin{aligned}
R&=6\alpha\frac{\overline{W}^2\nabla^2W^2}{1-2\alpha W^2\overline{W}^2}=6\alpha\overline{W}^2\nabla^2W^2+12\alpha^2\overline{W}^4W^2\nabla^2W^2\\
\overline{R}&=6\alpha\frac{W^2\overline{\nabla}^2\overline{W}^2}{1-2\alpha W^2\overline{W}^2}=6\alpha W^2\overline{\nabla}^2\overline{W}^2+12\alpha^2W^4\overline{W}^2\overline{\nabla}^2\overline{W}^2\,.
\end{aligned}
\tag{4.16}
$$

The $\chi^{A\dot{A}}$ field equation can also be immediately read:

$$
\begin{aligned}
\frac{G_{A\dot A}}{\alpha} = {}& 2G_{A\dot A}W^2\overline{W}^2 + \frac{i}{4}\overline{W}^2\nabla_{A\dot A}W^2 - \frac{i}{4}W^2\nabla_{A\dot A}\overline{W}^2 + \frac14\nabla_A\left(W^2\right)\nabla_{\dot A}\overline{W}^2 - \\
& -\Bigg[-\frac{i}{8}\left(\nabla^B\overline{R}\right)\nabla^C{}_{\dot A}\left(W_{ABC}\overline{W}^2\right) + \frac{3}{16}\left(\nabla^B\overline{R}\right)G^C{}_{\dot A}W_{ABC}\overline{W}^2 - \\
& \quad - \frac{9}{32}\nabla^B\left(\overline{R}G^C{}_{\dot A}W_{ABC}\overline{W}^2\right) + \frac{9}{16}\nabla^{\dot D}\left(G^B{}_{\dot A}G^C{}_{\dot D}W_{ABC}\overline{W}^2\right) + \\
& \quad + \frac{9}{16}\left(\nabla_{\dot A}\nabla^B - \nabla^B\nabla_{\dot A}\right)\nabla^{\dot D}\left(G^C{}_{\dot D}W_{ABC}\overline{W}^2\right) - \\
& \quad - \frac34\left(\nabla_{\dot D}\nabla^B - \nabla^B\nabla_{\dot D}\right)\left(\left(\nabla_{\dot A}G^{C\dot D}\right)W_{ABC}\overline{W}^2\right) + \\
& \quad + \frac98 i\nabla^{\dot D}\nabla^B{}_{\dot A}\left(G^C{}_{\dot D}W_{ABC}\overline{W}^2\right) + \frac{3}{16}i\nabla^B\left(\overline{R}\nabla^C{}_{\dot A}\left(W_{ABC}\overline{W}^2\right)\right) + \\
& \quad + \frac18\overline{R}\left(\nabla^BG^C{}_{\dot A}\right)W_{ABC}\overline{W}^2 + \frac{9}{16}i\nabla^{\dot D}\left(G^C{}_{\dot D}\nabla^B{}_{\dot A}\left(W_{ABC}\overline{W}^2\right)\right) - \\
& \quad - \frac{3}{16}i\nabla_{\dot A}\left(G^{B\dot D}\nabla^C{}_{\dot D}\left(W_{ABC}\overline{W}^2\right)\right) + \frac34\nabla^{\dot D}\nabla^B{}_{\dot A}\nabla^C{}_{\dot D}\left(W_{ABC}\overline{W}^2\right) - \\
& \quad - \frac{15}{16}i\nabla^{\dot D}\left(G^B{}_{\dot A}\nabla^C{}_{\dot D}\left(W_{ABC}\overline{W}^2\right)\right) - \\
& \quad - \frac38 i\left(\nabla_{\dot A}\nabla^B - \nabla^B\nabla_{\dot A}\right)\nabla^{\dot D}\nabla^C{}_{\dot D}\left(W_{ABC}\overline{W}^2\right) + \text{h.c.}\Bigg]. \qquad (4.17)
\end{aligned}
$$

This equation must be rewritten in a different form. Using the supercommutation relations and the solutions for torsions and curvatures in appendix A.2, most of its terms may be rewritten in such a way that they contain a minimal number of derivatives and an explicit dependence on $R, G_{A\dot A}, W_{ABC}$. The actual computations for this purpose are very heavy, and nothing special can be learnt from them. We present the intermediate results (the expansion of each term in (4.17)) in appendix C. Here we present only the final results. Replacing each expression of appendix C in (4.17), we obtain the following expanded field equation for $G_{A\dot A}$:

$$
\begin{aligned}
G_{A\dot A} = {}& \frac14\frac{\alpha}{1+\alpha W^2\overline{W}^2}\left(\nabla_AW^2\right)\nabla_{\dot A}\overline{W}^2 - \frac{\alpha}{1+\alpha W^2\overline{W}^2}\times \\
& \times\Bigg[iW^2\nabla_{A\dot A}\overline{W}^2 + \frac32\left(\nabla^{\dot D}\nabla^B{}_{\underline{\dot A}}\nabla^C{}_{\underline{\dot D}}\overline{W}^2\right)W_{ABC} + \\
& \quad + \frac38 W_A{}^{BC}\left(\nabla^DW_{DBC}\right)\nabla_{\dot A}\overline{W}^2 - \\
& \quad - \frac{i}{8}\overline{W}^2\left(\nabla^B\nabla^C{}_{\dot A}\overline{R}\right)W_{ABC} - \frac{i}{8}\overline{W}^2\left(\nabla^C{}_{\dot A}\overline{R}\right)\nabla^BW_{ABC} - \\
& \quad - \frac{i}{8}\left(\nabla^B\overline{R}\right)\overline{W}^2\nabla^C{}_{\dot A}W_{ABC} - \frac{9}{32}i\left(\nabla^B\overline{R}\right)\left(\nabla^C{}_{\dot A}\overline{W}^2\right)W_{ABC} - \\
& \quad - \frac{1}{16}\overline{W}^2G^C{}_{\dot A}\left(\nabla^B\overline{R}\right)W_{ABC} + \frac{1}{32}\left(\nabla^C\overline{R}\right)\left(\nabla_{\dot A}\overline{W}^2\right)\nabla^BW_{ABC} - \\
& \quad - \frac{3}{16}i\overline{R}\left(\nabla^C{}_{\dot A}\overline{W}^2\right)\nabla^BW_{ABC} - \frac{1}{16}\overline{R}\overline{W}^2\left(\nabla^BG^C{}_{\dot A}\right)W_{ABC} -
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{i}{8}\overline{RW}^2\nabla^B\nabla^C{}_{\dot{A}}W_{ABC}-\frac{1}{16}\overline{RW}^2G^C{}_{\dot{A}}\nabla^BW_{ABC}+\\
&+\frac{3}{2}i\left(\nabla^B{}_{\underline{\dot{A}}}G^C{}_{\underline{\dot{D}}}\right)\left(\nabla^{\dot{D}}\overline{W}^2\right)W_{ABC}-\\
&-\frac{3}{4}\left(\nabla^B\nabla_{\underline{\dot{A}}}G^C{}_{\underline{\dot{D}}}\right)\left(\nabla^{\dot{D}}\overline{W}^2\right)W_{ABC}-\\
&-\frac{9}{2}i\left(\nabla^{B\dot{D}}\overline{W}^2\right)\left(\nabla_{\underline{\dot{A}}}G^C{}_{\underline{\dot{D}}}\right)W_{ABC}+\\
&+\frac{3}{2}\left(\nabla^{\dot{D}}\overline{W}^2\right)\left(\nabla_{\underline{\dot{A}}}G^C{}_{\underline{\dot{D}}}\right)\nabla^BW_{ABC}+\\
&+\frac{3}{16}\left(\overline{\nabla}^2\overline{W}^2\right)\left(\nabla^BG^C{}_{\dot{A}}\right)W_{ABC}-\frac{9}{2}G^B{}_{\underline{\dot{A}}}G^C{}_{\underline{\dot{D}}}\left(\nabla^{\dot{D}}\overline{W}^2\right)W_{ABC}-\\
&-3iG^{B\dot{D}}\left(\nabla_{\underline{\dot{A}}}\overline{W}^2\right)\nabla^C{}_{\underline{\dot{D}}}W_{ABC}+\frac{3}{2}iG^B{}_{\dot{A}}\left(\nabla^{\dot{D}}\overline{W}^2\right)\nabla^C{}_{\dot{D}}W_{ABC}-\\
&-\frac{9}{2}iG^{B\dot{D}}\left(\nabla_{\underline{\dot{A}}}\nabla^C{}_{\underline{\dot{D}}}\overline{W}^2\right)W_{ABC}+3iG^B{}_{\dot{A}}\left(\nabla^{\dot{D}}\nabla^C{}_{\dot{D}}\overline{W}^2\right)W_{ABC}+\\
&+\frac{3}{2}\left(\nabla^{\dot{D}}\nabla^C{}_{\dot{D}}\overline{W}^2\right)\nabla^B{}_{\dot{A}}W_{ABC}-\frac{3}{8}i\left(\overline{\nabla}^2\overline{W}^2\right)\nabla^B\nabla^C{}_{\dot{A}}W_{ABC}-\\
&-\frac{3}{8}i\left(\overline{\nabla}^2\nabla^B{}_{\dot{A}}\overline{W}^2\right)\nabla^CW_{ABC}+\text{h.c.}\Bigg]\,. \qquad (4.18)
\end{aligned}
$$

This is a complicated nonlinear differential equation for $G_{A\dot{A}}$, which cannot be solved in a closed form. The best we can do is to solve it as a power series in α. The main question is: does that expansion stop for some power of α, like in R, or is it an infinite expansion?

To answer this question, we must look at the different terms of (4.18). Expanding the overall factor

$$
\frac{\alpha}{1+\alpha W^2\overline{W}^2}=\alpha-\alpha^2W^2\overline{W}^2+\alpha^3W^4\overline{W}^4 \qquad (4.19)
$$

and keeping in mind that $W^5=0$, we see that it is enough to multiply (4.19) by a term with a free W_{ABC} factor to be sure that the expansion stops at α^2, like R. Therefore, if each and every term of (4.18) has, after each iteration, a free W_{ABC} factor (or something from which it can be extracted, like *one* derivative of W^2), we have a finite polynomial expansion of $G_{A\dot{A}}$. If at least one of the terms of (4.18) does not have such a factor, the iteration process will never stop and we will have an infinite expansion.

The problems with getting a free W_{ABC} factor after iteration arise from the terms with derivatives of $G_{A\dot{A}}$, because these terms may contribute with derivatives of W_{ABC}, which do not stop the iteration process.

Let's then start by analyzing each term of (4.18) which does *not* include derivatives of $G_{A\dot{A}}$. By inspection, we conclude that all of them have a W_{ABC} factor with the exception of the three last terms which are only proportional to ∇^CW_{ABC} or its derivatives. Therefore, using (A.25), we can write them as derivatives of $G_{A\dot{A}}$ plus terms proportional to W_{ABC}:

$$\begin{aligned}\nabla^B{}_{\dot A} W_{ABC} &= \frac{i}{2}\nabla_{\dot A}\nabla^B W_{ABC} + \frac{5}{2} i G^B{}_{\dot A} W_{ABC} \\ &= \nabla_{\dot A}\nabla_{\underline{A}}{}^{\dot B} G_{\underline{C}\dot B} + \frac{5}{2} i G^B{}_{\dot A} W_{ABC} \qquad (4.20)\\ \nabla^B\nabla^C{}_{\dot A} W_{ABC} &= -2i\nabla^B{}_{\dot A}\nabla_{\underline{A}}{}^{\dot B} G_{\underline{B}\dot B} + G^B{}_{\dot A}\nabla_{\underline{A}}{}^{\dot B} G_{\underline{B}\dot B} + \\ &\quad + \frac{11}{2} i \left(\nabla^B G^C{}_{\dot A}\right) W_{ABC}\,. \qquad (4.21)\end{aligned}$$

We have two kinds of terms to iterate in (4.18): the ones without and the ones with derivatives of $G_{A\dot A}$. Replacing (4.20) and (4.21) in (4.18), we conclude that the former terms all have a W_{ABC} factor after iteration and only have a finite contribution to the α expansion of $G_{A\dot A}$.

The terms of (4.18) with derivatives of $G_{A\dot A}$ are linearly independent and cannot be simplified. Each time we iterate a solution for $G_{A\dot A}$ of a certain order in α on each of these terms, we get terms with higher derivatives of W_{ABC} and no new factors of W_{ABC} itself. This means that, because of these terms, the actual solution for $G_{A\dot A}$ is, as opposite to R, an infinite series in α, with derivatives of W_{ABC} to all orders.

From what was mentioned in section 3, this result was expected. The nonalgebraic field equation for the auxiliary field A_m, because of its higher-derivative terms in the component action, is actually obtained by taking the $\theta = 0$ component of (4.18).

4.3 Computation of $G_{A\dot A}$ to second order in α

From the results of the previous subsection, we conclude that the complete on-shell supersymmetrization of $\mathcal{R}^4$ requires an infinite number of terms. In practice, what we do is to solve (4.18) for $G_{A\dot A}$ perturbatively order by order in α, by iterating, for n-th order, the $n-1$-th solution.

In this subsection, we solve (4.18) for $G_{A\dot A}$ up to second order in α (the order at which R stops). We must then first solve for $G_{A\dot A}$ to first order in α.

In order to do that first we take, in (4.18), only the terms which are of first order in α. We recall that, to order 0 in α (pure supergravity), both $G_{A\dot A} = 0$ and $R = 0$. The first non-trivial order is, then, α, which means that both $G_{A\dot A}$ and R necessarily contribute with, at least, one power of α. We also recall that, due to the off-shell identity (A.25), terms like $\nabla^C W_{ABC}$ are, at least, of order α, and therefore

$$\nabla_A W_{BCD} = \nabla_{\underline{A}} W_{\underline{BCD}} + \mathcal{O}(\alpha)\,. \qquad (4.22)$$

We then get

$$\begin{aligned}G_{A\dot A} = &-i\alpha W^2\nabla_{A\dot A}\overline{W}^2 + i\alpha\overline{W}^2\nabla_{A\dot A}W^2 + \frac{\alpha}{4}\left(\nabla_A W^2\right)\nabla_{\dot A}\overline{W}^2 - \\ &-\frac{3}{2}\alpha\left(\nabla^B\nabla_{\underline{A}}{}^{\dot B}\nabla_{\underline{B}}{}^{\dot C}W^2\right)W_{\dot A\dot B\dot C} - \\ &-\frac{3}{2}\alpha\left(\nabla^{\dot B}\nabla^B{}_{\underline{\dot A}}\nabla^C{}_{\underline{\dot B}}\overline{W}^2\right)W_{ABC} + \mathcal{O}\left(\alpha^2\right) \qquad (4.23)\end{aligned}$$

which satisfies

$$\nabla^{\dot{A}} G_{A\dot{A}} = -\frac{1}{4}\alpha \left(\nabla_A W^2\right) \overline{\nabla}^2 \overline{W}^2 - i\alpha W^2 \nabla^{\dot{A}} \nabla_{A\dot{A}} \overline{W}^2 + \mathcal{O}\left(\alpha^2\right). \tag{4.24}$$

We now do the same with (4.16):

$$\overline{R} = 6\alpha W^2 \overline{\nabla}^2 \overline{W}^2 + \mathcal{O}\left(\alpha^2\right). \tag{4.25}$$

By taking

$$\nabla_A \overline{R} = 6\alpha \left(\nabla_A W^2\right) \overline{\nabla}^2 \overline{W}^2 + 24 i\alpha W^2 \nabla^{\dot{A}} \nabla_{A\dot{A}} \overline{W}^2 + \mathcal{O}\left(\alpha^2\right) \tag{4.26}$$

we see that the off-shell relation (A.24) is satisfied to first order in α, as it should.

We now proceed keeping, from (4.18), only the terms which are of order α^2. These terms are:

$$\begin{aligned} G_{A\dot{A}} = {} & -\alpha W^2 \overline{W}^2 G_{A\dot{A}} + \frac{\alpha}{4}\left(\nabla_A W^2\right) \nabla_{\dot{A}} \overline{W}^2 + \\ & + \Bigg[i\alpha \overline{W}^2 \nabla_{A\dot{A}} W^2 - \frac{3}{2}\alpha \left(\nabla^{\dot{B}} \nabla^{B}{}_{\underline{\dot{A}}} \nabla^{C}{}_{\underline{\dot{B}}} \overline{W}^2\right) W_{ABC} - \\ & \quad - \frac{3}{8}\alpha \left(\nabla^D W_D{}^{BC}\right) W_{ABC} \nabla_{\dot{A}} \overline{W}^2 + \frac{i}{8}\alpha \overline{W}^2 \left(\nabla^B \nabla^C{}_{\dot{A}} \overline{R}\right) W_{ABC} + \\ & \quad + \frac{9}{32} i\alpha \left(\nabla^B \overline{R}\right) \left(\nabla^C{}_{\dot{A}} \overline{W}^2\right) W_{ABC} - \frac{3}{2} i\alpha \left(\nabla^B{}_{\underline{\dot{A}}} G^C{}_{\underline{\dot{D}}}\right) \left(\nabla^{\dot{D}} \overline{W}^2\right) W_{ABC} + \\ & \quad + \frac{3}{4}\alpha \left(\nabla^B \nabla_{\underline{\dot{A}}} G^C{}_{\underline{\dot{D}}}\right) \left(\nabla^{\dot{D}} \overline{W}^2\right) W_{ABC} - \\ & \quad - \frac{3}{16}\alpha \left(\overline{\nabla}^2 \overline{W}^2\right) \left(\nabla^B G^C{}_{\dot{A}}\right) W_{ABC} + \frac{9}{2} i\alpha G^{B\dot{D}} \left(\nabla_{\underline{\dot{A}}} \nabla^C{}_{\underline{\dot{D}}} \overline{W}^2\right) W_{ABC} - \\ & \quad - 3i\alpha G^B{}_{\dot{A}} \left(\nabla^{\dot{D}} \nabla^C{}_{\dot{D}} \overline{W}^2\right) W_{ABC} - \frac{3}{2}\alpha \left(\nabla^{\dot{D}} \nabla^C{}_{\dot{D}} \overline{W}^2\right) \nabla^B{}_{\dot{A}} W_{ABC} + \\ & \quad + \frac{9}{2} i\alpha \left(\nabla^{B\dot{D}} \overline{W}^2\right) \left(\nabla_{\underline{\dot{A}}} G^C{}_{\underline{\dot{D}}}\right) W_{ABC} + \frac{3}{8} i\alpha \left(\overline{\nabla}^2 \overline{W}^2\right) \nabla^B \nabla^C{}_{\dot{A}} W_{ABC} + \\ & \quad + \frac{3}{8} i\alpha \left(\overline{\nabla}^2 \nabla^B{}_{\dot{A}} \overline{W}^2\right) \nabla^C W_{ABC} + \text{h.c.} \Bigg] + \mathcal{O}\left(\alpha^3\right). \end{aligned} \tag{4.27}$$

We must now replace the solutions (4.23) and (4.25) for $G_{A\dot{A}}$ and R in (4.27). For that, we need a series of intermediate expressions which we present in appendix D. The actual computations are very heavy, and nothing special can be learnt from them. Here we present only the final results. Replacing each expression of appendix D in (4.27), we obtain the following expanded field equation for $G_{A\dot{A}}$:[5]

$$G_{A\dot{A}} = \frac{1}{4}\alpha \left(\nabla_A W^2\right) \nabla_{\dot{A}} \overline{W}^2 - \frac{1}{4}\alpha^2 W^2 \overline{W}^2 \left(\nabla_A W^2\right) \nabla_{\dot{A}} \overline{W}^2 +$$

---

[5] When we double-underline a pair of indices, like B, C in $W^{\underline{\underline{B}}\underline{D}E} W^{\underline{\underline{C}}}{}_{E\underline{F}}$, we mean that we are symmetrizing that pair of indices and there is another pair of indices of the same type (D, F in our case) which is being symmetrized independently of the first.

$$
\begin{aligned}
&+\Big[i\alpha\overline{W}^2\nabla_{A\dot A}W^2 - \frac{3}{2}\alpha\left(\nabla^{\dot B}\nabla^{B}{}_{\dot B}\nabla^{C}{}_{\dot A}\overline{W}^2\right)W_{ABC} - \\
&\quad - \frac{3}{8}\alpha\left(\nabla^{D}W_{D}{}^{BC}\right)W_{ABC}\nabla_{\dot A}\overline{W}^2 - \frac{3}{2}\alpha\left(\nabla^{\dot D}\nabla^{C}{}_{\dot D}\overline{W}^2\right)\nabla^{B}{}_{\dot A}W_{ABC} + \\
&\quad + \frac{3}{8}i\alpha\left(\overline{\nabla}^2\overline{W}^2\right)\nabla^{B}\nabla^{C}{}_{\dot A}W_{ABC} + \frac{3}{8}i\alpha\left(\overline{\nabla}^2\nabla^{B}{}_{\dot A}\overline{W}^2\right)\nabla^{C}W_{ABC} - \\
&\quad - 2i\alpha^2W^4\overline{W}^2\nabla_{A\dot A}W^2 - \frac{3}{2}\alpha^2W^2\overline{W}^2\left(\nabla^{\dot B}\nabla^{B}{}_{\dot B}\nabla^{C}{}_{\dot A}\overline{W}^2\right)W_{ABC} + \\
&\quad + 9\alpha^2W^2\left(\nabla^{B\dot B}\overline{W}^2\right)\left(\nabla_{\underline{\dot A}}\nabla^{C}{}_{\underline{\dot B}}\overline{W}^2\right)W_{ABC} - \\
&\quad - \frac{39}{4}\alpha^2W^2\left(\nabla^{B}{}_{\dot A}\overline{W}^2\right)\left(\nabla^{\dot D}\nabla^{C}{}_{\dot D}\overline{W}^2\right)W_{ABC} - \\
&\quad - 3\alpha^2W^2\left(\nabla^{B}{}_{\underline{\dot A}}\nabla^{C}{}_{\underline{\dot B}}\overline{W}^2\right)\left(\nabla^{\dot B}\overline{W}^2\right)W_{ABC} + \\
&\quad + 18\alpha^2W^{BDE}W^{C}{}_{EF}W_{ABC}\left(\nabla_{\underline{D\dot A}}\nabla^{\underline{F}}{}_{\underline{\dot B}}\overline{W}^2\right)\nabla^{\dot B}\overline{W}^2 - \\
&\quad - \frac{9}{2}i\alpha^2\left(\nabla^{\dot C}\nabla^{\underline{B}}{}_{\dot C}\nabla^{\underline{E}}{}_{\underline{\dot A}}\nabla^{\underline{F}}{}_{\underline{\dot B}}\overline{W}^2\right)W^{C}{}_{EF}W_{ABC}\nabla^{\dot B}\overline{W}^2 - \\
&\quad - \frac{9}{32}i\alpha^2\left(\overline{\nabla}^2\overline{W}^2\right)\left(\nabla_{A\dot A}\overline{W}^2\right)W^{FDE}W_{F}{}^{BC}\nabla_{\underline{B}}W_{\underline{CDE}} + \\
&\quad + \frac{27}{8}i\alpha^2W^{EFC}W_{ABC}\left(\nabla^{B\dot B}\overline{W}^2\right)\overline{\nabla}^2\nabla_{E\underline{\dot A}}\nabla_{F\underline{\dot B}}\overline{W}^2 - \\
&\quad - \frac{27}{4}i\alpha^2W^{BDE}W_{ABC}\left(\nabla^{\dot C}\nabla_{D\dot C}\nabla_{E\dot B}\overline{W}^2\right)\nabla_{\underline{\dot A}}\nabla^{C\underline{\dot B}}\overline{W}^2 - \\
&\quad - \frac{9}{2}i\alpha^2W^{BDE}W_{ABC}\left(\nabla^{\dot B}\nabla_{D\dot B}\nabla_{E\dot A}\overline{W}^2\right)\nabla^{\dot C}\nabla^{C}{}_{\dot C}\overline{W}^2 - \\
&\quad - 9\alpha^2\left(\nabla^{\dot D}\overline{W}^2\right)W^{\dot E}{}_{\underline{\dot A}\underline{\underline{\dot B}}}W_{\dot E\underline{\underline{\dot C}}\underline{\dot D}}W_{ABC}\nabla^{B\dot B}\nabla^{C\dot C}W^2 + \\
&\quad + \frac{27}{2}\alpha^2\left(\nabla^{C\dot C}\overline{W}^2\right)W_{\dot A\dot B\dot C}W^{BEF}W_{ABC}\nabla_{E}\nabla_{F}{}^{\dot B}W^2 + \\
&\quad + \frac{3}{2}\alpha^2\overline{W}^2\left(\nabla^{\dot D}\overline{W}^2\right)W_{ABC}\nabla^{B}{}_{\underline{\dot A}}\nabla^{C}{}_{\underline{\dot B}}W^2 - \\
&\quad - \frac{9}{2}\alpha^2\overline{W}^2\left(\nabla_{\underline{\dot A}}\nabla^{B}{}_{\underline{\dot B}}\overline{W}^2\right)W_{ABC}\nabla^{C\dot B}W^2 - \\
&\quad - \frac{9}{16}i\alpha^2\overline{W}^2\left(\overline{\nabla}^2\overline{W}^2\right)W_{ABC}\nabla^{B}\nabla^{C}{}_{\dot A}W^2 - \\
&\quad - \frac{3}{4}i\alpha^2\overline{W}^2\left(\overline{\nabla}^2\nabla^{C}{}_{\dot A}\overline{W}^2\right)W_{ABC}\nabla^{B}W^2 - \\
&\quad - \frac{9}{32}i\alpha^2\left(\overline{\nabla}^2\overline{W}^2\right)W^2W_{\dot A}{}^{\dot B\dot C}\nabla_{\dot B}\nabla_{A\dot C}\overline{W}^2 + \\
&\quad + \frac{9}{4}i\alpha^2\left(\nabla^{\dot B}\overline{W}^2\right)W_{\underline{\dot B}\dot D\dot E}W_{ABC}\nabla^{E}\nabla_{E\underline{\dot A}}\nabla^{B\dot D}\nabla^{C\dot E}W^2 - \\
&\quad - \frac{9}{4}i\alpha^2\left(\nabla^{\dot B}\overline{W}^2\right)W_{\dot A\dot B\dot C}W_{ABC}\nabla^{B}\nabla^{D\dot D}\nabla_{D\dot D}\nabla^{C\dot C}W^2 + \\
&\quad + \frac{9}{2}i\alpha^2\left(\nabla^{\dot B}\overline{W}^2\right)\left(\nabla_{\underline{\dot A}}W_{\underline{\dot B\dot D\dot E}}\right)W^{\dot D\dot E\dot C}W_{ABC}\nabla^{B}\nabla^{C}{}_{\dot C}W^2 +
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{9}{4}i\alpha^2\left(\nabla^{\dot{B}}\overline{W}^2\right)W_{\dot{A}\dot{B}\dot{C}}W_{ABC}\left(\nabla^{\underline{B}}W^{\underline{CDE}}\right)\nabla_D\nabla_E{}^{\dot{C}}W^2+\\
&+\frac{27}{4}\alpha^2\left(\nabla^{B\dot{B}}\overline{W}^2\right)W_{\dot{A}\dot{B}\dot{C}}W_{ABC}\nabla^{D\dot{D}}\nabla_{D\dot{D}}\nabla^{C\dot{C}}W^2-\\
&-\frac{9}{64}\alpha^2\left(\overline{\nabla}^2\overline{W}^2\right)W_{\dot{A}\dot{B}\dot{C}}W_{ABC}\nabla^2\nabla^{B\dot{B}}\nabla^{C\dot{C}}W^2+\\
&+\frac{27}{4}i\alpha^2\left(\nabla_{\underline{\dot{A}}}\nabla^B{}_{\underline{\dot{B}}}\overline{W}^2\right)W_{\dot{D}}{}^{\dot{B}\dot{C}}W_{ABC}\nabla^D\nabla_{D\dot{C}}\nabla^{B\dot{D}}W^2+\\
&+\frac{9}{2}i\alpha^2\left(\nabla^{\dot{D}}\nabla^B{}_{\dot{D}}\overline{W}^2\right)W_{\dot{A}\dot{B}\dot{C}}W_{ABC}\nabla^D\nabla_D{}^{\dot{C}}\nabla^{B\dot{B}}W^2-\\
&-\frac{27}{2}\alpha^2\left(\nabla^{B\dot{B}}\overline{W}^2\right)\left(\nabla_{\underline{\dot{A}}}W_{\underline{\dot{B}\dot{C}\dot{D}}}\right)W^{\dot{C}\dot{D}\dot{E}}W_{ABC}\nabla^C{}_{\dot{E}}W^2+\\
&+\frac{9}{16}i\alpha^2W^2\left(\nabla^{\dot{B}}\overline{W}^2\right)\nabla_{\underline{\dot{A}}}\nabla^{D\dot{D}}\nabla_{D\dot{D}}\nabla_{A\underline{\dot{B}}}\overline{W}^2+\\
&+\frac{9}{8}i\alpha^2W^2\left(\nabla^{\dot{B}}\overline{W}^2\right)\left(\nabla_{\underline{\dot{A}}}W_{\underline{\dot{B}\dot{C}\dot{D}}}\right)\nabla^{\dot{D}}\nabla_A{}^{\dot{C}}\overline{W}^2-\\
&-\frac{9}{64}i\alpha^2W^2\left(\overline{\nabla}^2\overline{W}^2\right)\nabla^{D\dot{D}}\nabla_{D\dot{D}}\nabla_{A\dot{A}}\overline{W}^2-\\
&-\frac{3}{2}i\alpha^2\left(\overline{\nabla}^2\overline{W}^2\right)\left(\nabla^C{}_{\dot{A}}\overline{W}^2\right)W_{ABC}\nabla^BW^2-\\
&-\frac{27}{16}i\alpha^2\left(\nabla^{\dot{B}}\overline{W}^2\right)\left(\nabla_{\dot{B}}\overline{W}^2\right)W_{ABC}\nabla^B\nabla^C{}_{\dot{A}}W^2-\\
&-\frac{9}{4}i\alpha^2\left(\nabla^{\dot{B}}\overline{W}^2\right)\left(\nabla_{\underline{\dot{A}}}\nabla^B{}_{\underline{\dot{B}}}\overline{W}^2\right)W_{ABC}\nabla^BW^2+\\
&+\frac{9}{2}i\alpha^2\left(\nabla^{\dot{B}}\overline{W}^2\right)\left(\nabla^B{}_{\underline{\dot{A}}}W_{\underline{\dot{B}\dot{C}\dot{D}}}\right)W_{ABC}\nabla^D\nabla_D{}^{\dot{D}}\nabla^{C\dot{C}}W^2-\\
&-\frac{9}{8}i\alpha^2\left(\nabla^{\dot{B}}\overline{W}^2\right)\left(\nabla^{\dot{D}}\nabla_{D\dot{D}}\nabla_{E\dot{A}}\overline{W}^2\right)W_{ABC}\nabla^{\underline{B}}{}_{\dot{B}}W^{\underline{CDE}}-\\
&-\frac{15}{4}\alpha^2\left(\nabla^{\dot{B}}\overline{W}^2\right)\left(\nabla^B{}_{\underline{\dot{B}}}\overline{W}^2\right)W_{ABC}\nabla^C{}_{\underline{\dot{A}}}W^2+\\
&+\frac{9}{16}\alpha^2\left(\nabla^{\dot{B}}\overline{W}^2\right)\left(\overline{\nabla}^2\nabla_{D\underline{\dot{B}}}\nabla_{E\underline{\dot{A}}}\overline{W}^2\right)W_{ABC}\nabla^{\underline{B}}W^{\underline{CDE}}-\\
&-\frac{27}{4}i\alpha^2\left(\nabla^{B\dot{B}}\overline{W}^2\right)\left(\nabla_{\underline{\dot{A}}}W_{\underline{\dot{B}\dot{C}\dot{D}}}\right)W_{ABC}\nabla^D\nabla_D{}^{\dot{D}}\nabla^{C\dot{C}}W^2-\\
&-\frac{9}{32}\alpha^2\left(\nabla^{\dot{B}}\nabla_{D\dot{B}}\nabla_{E\dot{A}}\overline{W}^2\right)\left(\overline{\nabla}^2\overline{W}^2\right)W_{ABC}\nabla^{\underline{B}}W^{\underline{CDE}}+\\
&+\frac{3}{8}i\alpha^2\left(\nabla_{\dot{A}}\overline{W}^2\right)\left(\nabla^{\dot{B}}\nabla^B{}_{\dot{B}}\overline{W}^2\right)W_{ABC}\nabla^CW^2-\\
&-\frac{9}{8}i\alpha^2\left(\nabla^{\dot{B}}\overline{W}^2\right)\left(\nabla_{\underline{\dot{A}}}\nabla^B{}_{\underline{\dot{B}}}\overline{W}^2\right)W_{ABC}\nabla^CW^2+\\
&+\frac{27}{8}\alpha^2\left(\nabla^{B\dot{B}}\overline{W}^2\right)\left(\nabla_{\dot{A}}\overline{W}^2\right)W_{ABC}\nabla^C{}_{\dot{B}}W^2+\text{h.c.}\Bigg]+\mathcal{O}\left(\alpha^3\right).
\end{aligned}
\tag{4.28}
$$

Substituting (4.28) and the exact expression (4.16) in the expressions of appendix A.2, we get the solution of the superspace torsions and curvatures to second order in α. The knowledge of these expressions (though (4.28) being very complicated) may be relevant in order to check consistency with higher dimensional results, such as the on-shell solution of Bianchi identities or the identification of auxiliary fields and constraints such as (A.24).

5. Discussion, conclusions and future perspectives

In this paper, we have shown that, in order to supersymmetrize $\mathcal{R}^4$ on shell, one must introduce an infinite number of terms. We derived this result in four dimensions, but it is valid in higher dimensions. Any dimensional reduction of an equivalent higher-dimensional result must agree with it. Its infinite number of terms shows that the complete supersymmetric $\mathcal{R}^4$ theory is nonlocal.

By being able to get a complete solution for R (4.16), we see that we can put our theory only partially off-shell (by eliminating the auxiliary fields M, N and leaving A_m) with just a finite number of terms.

The result we obtained is not unexpected: since the late 80s, during the supersymmetrization of the Lorentz Chern-Simons term, people obtained nonlinear differential equations for auxiliary fields [9, 11, 12]. The same was valid for the four-dimensional case [17].

In [12], auxiliary fields Y_{mnpq} and S_{mnp} were introduced while solving the Bianchi identities. These fields had to be expanded to an infinite series in a perturbative parameter c_2 (defined below). What is remarkable is that a complete, all order solution to Y_{mnpq} exists (but depending on S_{mnp}). It is then possible to eliminate Y_{mnpq}, leaving only S_{mnp}. This situation is similar to the one we saw in this paper: the auxiliary fields Y_{mnpq} and S_{mnp} are analogous to our R and A_m, respectively.

There is, anyway, an important difference between working in ten or eleven dimensions and the four dimensional case we considered in this paper. In ten dimensional type I supergravity in superspace [33], one must introduce a 2-form B_{MN}, with field strength H_{MNP}, in order to accomodate the x-space 2-form B_{mn}. The field strength H satisfies the Bianchi identity $dH = 0$ (in differential form notation) which, together with the torsion Bianchi identities, form a set of coupled equations to be solved simultaneously. If we consider type I supergravity coupled to super Yang-Mills [34], we have an additional one-form A_M with field strength F_{MN} and Bianchi identity $dF = 0$, but we must modify the H Bianchi identity by $dH = \mathrm{tr}F^2$, because of the corresponding redefinition of the x-space field strength [35]. In eleven-dimensional supergravity there is a similar situation [36]: one needs to introduce a superspace 3-form X_{MNP} in order to accomodate the x-space 3-form X_{mnp}, but its field strength Y_{MNPQ} satisfies the Bianchi identity $dY = 0$. In all these cases, even if we had an eventual off-shell formalism, we would still have a system of coupled nonlinear superspace Bianchi identities which would have to be solved by an iterative procedure.[6]

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[6] If we instead consider anomaly-free ten-dimensional supergravity, there is an extra complication:

In four dimensional supergravity, because of its smaller field content, we only need to consider the torsion Bianchi identities. As it is well known, there is a complete off-shell solution to them. The non-linearities may only show up when we go on-shell, as we did with a higher-derivative action. This was the reason for the need of an infinite number of terms in a completely supersymmetric action. From the results of this paper, it should be very clear that the necessity for an infinite number of terms in the corresponding problem in ten or eleven dimensions comes also from the fact that we are supersymmetrizing a higher-derivative theory, and not only because of coupled nonlinear Bianchi identities.

Knowing this, there are two alternatives to proceed supersymmetrizing $\mathcal{R}^4$ actions. One is to work order by order in some perturbative parameter, changing the supersymmetry transformations in x-space. This is what has been done in [14] in ten dimensions and in [15] in eleven. The second alternative is to introduce auxiliary fields. They are easier to identify while solving the Bianchi identities in superspace, where one can relax the torsion constraints. This procedure has been followed by [16, 37, 38].

From the results we got in this paper, we can compare our solution to the superspace Bianchi identities with higher dimensional ones, after eliminating auxiliary fields in these solutions. A comparison of the solutions before and after the elimination of the auxiliary fields could clarify their role and possible identification.

We can also expand the action (3.3) in components and eliminate the auxiliary fields M, N, getting a partially off-shell supersymmetric x-space $\mathcal{R}^4$ action, which should be easier to compare with the $\mathcal{R}^4$ superinvariants constructed in higher dimensions. This work is in progress [28].

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the Green-Schwarz mechanism requires an extra redefinition of the x-space three-form field strength, as we mentioned, which implies an extra term in $dH = \mathrm{tr}(c_1 F^2 - c_2 R^2)$. This difficulty can be remedied with a theorem proved in [11], which reduces this case to the previous one ($dH = \mathrm{tr} F^2$) through a superfield redefinition.

A. Survey of known results and conventions

A.1 General conventions

We work in flat Minkowski space-time with the metric $\eta_{mn} = \mathrm{diag}\,(-1,1,1,1)$. The Levi-Civita tensor is given by $\varepsilon^{0123} = -\varepsilon_{0123} = 1$.

In flat space the vector indices are $m, n, \ldots$, the two-component spinor indices are $A, \dot{A}, \ldots$, and the superindices are $M, N, \ldots$ In curved space, the vector indices are $\mu, \nu, \ldots$ and the superindices are $\Lambda, \Pi, \ldots$ We do not use curved spinor indices in x-space.

We raise and lower two-component spinor indices with the tensors

$$\varepsilon_{AB} = \varepsilon^{AB} = -\varepsilon_{\dot{A}\dot{B}} = -\varepsilon^{\dot{A}\dot{B}}\,, \qquad \varepsilon_{12} = 1\,. \tag{A.1}$$

The contractions of spinor indices *always* follow the north-west rule (the same is valid for superindices in general). We then define

$$\theta^2 = \theta^A\theta_A = \varepsilon^{AB}\theta_B\theta_A\,, \qquad \bar{\theta}^2 = \theta^{\dot{A}}\theta_{\dot{A}} = \varepsilon^{\dot{A}\dot{B}}\theta_{\dot{B}}\theta_{\dot{A}}\,. \tag{A.2}$$

The decomposition of a tensor with two spinor indices on its symmetric (underlined) and antisymmetric (trace) parts is always (for dotted/undotted, upper/lower indices) given by

$$T_{AB} = T_{\underline{AB}} - \frac{1}{2}\varepsilon_{AB}T^C{}_C\,, \qquad T_{\dot{A}\dot{B}} = T_{\underline{\dot{A}\dot{B}}} - \frac{1}{2}\varepsilon_{\dot{A}\dot{B}}T^{\dot{C}}{}_{\dot{C}}\,. \tag{A.3}$$

Dotted and undotted spinor indices are related through complex conjugation (that's what a bar means):

$$\overline{\theta_A} = \theta_{\dot{A}}\,, \qquad \overline{\theta^A} = -\theta^{\dot{A}}\,. \tag{A.4}$$

For fermionic derivatives, the rules are

$$\overline{\left(\frac{\partial}{\partial\theta_A}\right)} = -\frac{\partial}{\partial\theta_{\dot{A}}}\,, \; \overline{\left(\frac{\partial}{\partial\theta^A}\right)} = \frac{\partial}{\partial\theta^{\dot{A}}}\,. \tag{A.5}$$

We define (as proposed in [26] — notice that our conventions differ from this book)

$$A_{A\dot{A}} = A_m\sigma^m{}_{A\dot{A}}\,, \qquad A_m = \frac{1}{2}A_{A\dot{A}}\sigma_m{}^{A\dot{A}} \tag{A.6}$$

with

$$\sigma^m{}_{A\dot{A}} = (I, \vec{\sigma})_{A\dot{A}}\,, \qquad \sigma_m{}^{A\dot{A}} = (-I, \vec{\sigma})^{A\dot{A}} \tag{A.7}$$

$\vec{\sigma}$ are the three Pauli matrices.

We write (in superspace, at least) all the vector indices contracted like this. Because of the Lorentz invariance of $\sigma^m{}_{A\dot{A}}$, we can use in our superspace calculations with these "contracted vector indices" the normal rules for the spinor indices.

A.2 Superspace conventions

The superspace Lorentz covariant derivative is given by

$$\nabla_\Lambda = \partial_\Lambda + \frac{1}{2}\Omega_\Lambda{}^{mn} J_{mn}\,, \qquad \nabla_M = E_M{}^\Lambda \nabla_\Lambda\,. \tag{A.8}$$

Our definitions of supertorsion and (Lorentz-valued) supercurvature come from

$$[\nabla_M, \nabla_N\} = T_{MN}{}^R \nabla_R + \frac{1}{2} R_{MN}{}^{rs} J_{rs}\,. \tag{A.9}$$

Explicitly, we have

$$T_{MN}{}^R = E_M{}^\Lambda \partial_\Lambda E_N{}^\Pi E_\Pi{}^R + \Omega_{MN}{}^R - (-)^{MN}\,(M \leftrightarrow N) \tag{A.10}$$

$$R_{MN}{}^{rs} = E_M{}^\Lambda E_N{}^\Pi \left\{ \partial_\Lambda \Omega_\Pi^{\ rs} + \Omega_\Lambda^{\ rk} \Omega_{\Pi k}{}^s - (-)^{\Lambda\Pi}\,(\Lambda \leftrightarrow \Pi) \right\}. \tag{A.11}$$

Our choice of constraints is that all the torsions which are missing in the following list are set equal to zero, with the exception of

$$T_{A\dot{B}}{}^{\ m} = -2i\sigma^m_{A\dot{B}}\,. \tag{A.12}$$

The solutions to the Bianchi identities [20] are, in our conventions, for the torsions:

$$T_{A\dot{A}\ \dot{B}C} = \frac{i}{12}\varepsilon_{AC}\varepsilon_{\dot{A}\dot{B}}\overline{R} \tag{A.13}$$

$$T_{A\dot{A}\ BC} = \frac{i}{4}\left(3\varepsilon_{AB}G_{C\dot{A}} + \varepsilon_{AC}G_{B\dot{A}} - 3\varepsilon_{BC}G_{A\dot{A}}\right) \tag{A.14}$$

$$\begin{aligned} T_{A\dot{A}\ B\dot{B}\ C} = {} & -\varepsilon_{\dot{A}\dot{B}}\left(W_{ABC} - \frac{1}{2}\varepsilon_{AC}\nabla^{\dot{C}}G_{B\dot{C}} - \frac{1}{2}\varepsilon_{BC}\nabla^{\dot{C}}G_{A\dot{C}}\right) - \\ & -\varepsilon_{AB}\left(\ \nabla_{\underline{\dot{A}}}G_{C\underline{\dot{B}}}\right). \end{aligned} \tag{A.15}$$

Obviously, complex-conjugating (A.13), (A.14), (A.15) we get the solutions for the complex-conjugated torsions. The same is valid for the curvatures, whose solutions to the Bianchi identities are:

$$R_{ABCD} = \frac{1}{6}\left(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC}\right)R \tag{A.16}$$

$$R_{A\dot{B}CD} = \varepsilon_{AC}G_{D\dot{B}} + \varepsilon_{AD}G_{C\dot{B}} \tag{A.17}$$

$$R_{\dot{A}\dot{B}CD} = 0 \tag{A.18}$$

$$\begin{aligned} R_{E\ A\dot{A}\ \dot{B}\dot{C}} = {} & -2i\varepsilon_{EA}W_{\dot{A}\dot{B}\dot{C}} - \frac{i}{96}\varepsilon_{EA}\varepsilon_{\dot{A}\dot{B}}\nabla_{\dot{C}}R - \frac{i}{96}\varepsilon_{EA}\varepsilon_{\dot{A}\dot{C}}\nabla_{\dot{B}}R + \\ & +\frac{i}{2}\varepsilon_{\dot{A}\dot{C}}\nabla_{\underline{E}}G_{\underline{A}\dot{B}} + \frac{i}{2}\varepsilon_{\dot{A}\dot{B}}\nabla_{\underline{E}}G_{\underline{A}\dot{C}} \end{aligned} \tag{A.19}$$

$$R_{E\ A\dot{A}\ BC} = -i\varepsilon_{EA}\nabla_{\underline{B}}G_{\underline{C}\dot{A}} - \frac{i}{2}\varepsilon_{EB}\nabla_{\underline{C}}G_{\underline{A}\dot{A}} - \frac{i}{2}\varepsilon_{EC}\nabla_{\underline{A}}G_{\underline{B}\dot{A}} -$$

$$
\begin{aligned}
&-\frac{i}{32}\left(\varepsilon_{EB}\varepsilon_{AC}+\varepsilon_{EC}\varepsilon_{AB}\right)\nabla_{\dot{A}}R && \text{(A.20)}\\
R_{A\dot{A}\ B\dot{B}\ CD} = &-\varepsilon_{\dot{A}\dot{B}}\nabla_{\underline{A}}W_{\underline{BCD}}-\\
&-\varepsilon_{\dot{A}\dot{B}}\left(\varepsilon_{AC}\varepsilon_{BD}+\varepsilon_{AD}\varepsilon_{BC}\right)\cdot\\
&\cdot\left(\frac{1}{192}\left(\nabla^2\overline{R}+\overline{\nabla}^2R\right)+\frac{1}{144}R\overline{R}+\frac{1}{4}G^{E\dot{E}}G_{E\dot{E}}\right)+\\
&+\varepsilon_{AB}\left[\nabla_{\underline{C}}\nabla_{\underline{\dot{A}}}G_{\underline{D\dot{B}}}-i\nabla_{\underline{C\dot{A}}}G_{\underline{D\dot{B}}}+G_{\underline{C\dot{A}}}G_{\underline{D\dot{B}}}\right]. && \text{(A.21)}
\end{aligned}
$$

The superfields $G_{A\dot{A}}$ and W_{ABC} have the following complex conjugation properties:

$$
\begin{aligned}
\overline{G_{A\dot{B}}} &= G_{B\dot{A}}\\
\overline{W_{ABC}} &= W_{\dot{A}\dot{B}\dot{C}}\,. && \text{(A.22)}
\end{aligned}
$$

R and $W_{\dot{A}\dot{B}\dot{C}}$ are antichiral:

$$
\begin{aligned}
\nabla_A R &= 0\\
\nabla_A W_{\dot{A}\dot{B}\dot{C}} &= 0\,. && \text{(A.23)}
\end{aligned}
$$

The following relations between R, $G_{A\dot{A}}$ and W_{ABC} are a consequence of the Bianchi identities:

$$
\nabla^A G_{A\dot{B}} = \frac{1}{24}\nabla_{\dot{B}}R \qquad \text{(A.24)}
$$

$$
\nabla^A W_{ABC} = i\left(\nabla_{B\dot{A}}G_C{}^{\dot{A}}+\nabla_{C\dot{A}}G_B{}^{\dot{A}}\right). \qquad \text{(A.25)}
$$

From (A.24) and its complex conjugate and the solution of the Bianchi identities, we may also derive the following useful relation between superfields:

$$
\nabla^2\overline{R}-\overline{\nabla}^2R = 96i\nabla^n G_n\,. \qquad \text{(A.26)}
$$

These relations (A.24), (A.25), (A.26) are off-shell identities (not field equations).

B. The constrained variation of W_{ABC}

In this appendix, we present the details of the computation of δW_{ABC}. From the solution of the Bianchi identities, we have

$$
W_{ABC} = \frac{1}{2}T_{\underline{A}}{}^{\dot{A}}{}_{\underline{B}\dot{A}\ \underline{C}}\,,\qquad W_{\dot{A}\dot{B}\dot{C}} = -\frac{1}{2}T^{A}{}_{\underline{\dot{A}}A\underline{\dot{B}\dot{C}}}\,. \qquad \text{(B.1)}
$$

Therefore, from (2.5) and the torsions (A.12), (A.13), (A.14) and (A.15), we have for the constrained variation

$$
\begin{aligned}
\delta W_{ABC} = &-H_{\underline{A}}{}^{\dot{A}\ D}T_{D\ \underline{B}\dot{A}\ \underline{C}}-\frac{1}{2}H_{\underline{A}}{}^{\dot{A}\ D\dot{D}}T_{D\dot{D}\ \underline{B}\dot{A}\ \underline{C}}+\\
&+\frac{1}{2}T_{\underline{A}}{}^{\dot{A}}{}_{\underline{B}\dot{A}}{}^{D}H_{D\underline{C}}+\frac{1}{2}T_{\underline{A}}{}^{\dot{A}}{}_{\underline{B}\dot{A}}{}^{\dot{D}}H_{\dot{D}\underline{C}}-\nabla_{\underline{A}}{}^{\dot{A}}H_{\underline{B}\dot{A}\ \underline{C}}\,. && \text{(B.2)}
\end{aligned}
$$

This expression needs to be simplified. For that, we use again our knowledge of the torsions from the solution of the Bianchi identities and the equations for $H_M{}^N$ we have just derived. We get for each term in (B.2)

$$\begin{aligned}
-H_{\underline{A}}{}^{\dot{A}\ D}T_{D\ \underline{B}\dot{A}\ \underline{C}} = {} & \frac{3}{32}i\overline{R}\left(\nabla_{\underline{A}}\chi_{\underline{B}}{}^{\dot{A}}\right)G_{\underline{C}\dot{A}} + \\
& + \frac{3}{16}i\left(\nabla^{\dot{A}}\left(\nabla^{\dot{E}}\nabla_{\underline{A}} - \nabla_{\underline{A}}\nabla^{\dot{E}}\right)\chi_{\underline{B}\dot{E}}\right)G_{\underline{C}\dot{A}} + \\
& + \frac{3}{8}\left(\nabla_{\underline{A}}{}^{\dot{E}}\nabla^{\dot{A}}\chi_{\underline{B}\dot{E}}\right)G_{\underline{C}\dot{A}} + \\
& + \frac{i}{16}\chi_{\underline{B}}{}^{\dot{A}}\left(\nabla_{\underline{A}}\overline{R}\right)G_{\underline{C}\dot{A}} - \frac{3}{16}iG_{\underline{A}\dot{E}}G_{\underline{C}\dot{A}}\nabla^{\dot{E}}\chi_{\underline{B}}{}^{\dot{A}}
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
H_{\underline{A}}{}^{\dot{A}\ D\dot{D}}T_{D\dot{D}\ \underline{B}\dot{A}\ \underline{C}} = {} & \left[\frac{2}{3}\left(\nabla^2 + \frac{1}{3}R\right)U + \frac{2}{3}\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\overline{U} + \right. \\
& \left. + \frac{4}{3}i\chi^{E\dot{E}}G_{E\dot{E}} + \frac{i}{3}\left(\nabla^{E}\nabla^{\dot{E}} - \nabla^{\dot{E}}\nabla^{E}\right)\chi_{E\dot{E}}\right]W_{ABC} + \\
& + \frac{i}{2}\left(\nabla_{\dot{A}}\nabla_{\underline{A}} - \nabla_{\underline{A}}\nabla_{\dot{A}}\right)\chi^{D\dot{A}}W_{D\underline{BC}} - \\
& - \frac{i}{96}\left[\left(\nabla^{\dot{A}}\nabla_{\underline{A}} - \nabla_{\underline{A}}\nabla^{\dot{A}}\right)\chi_{\underline{C}\dot{A}}\right]\nabla_{\underline{B}}\overline{R} + \\
& + \frac{i}{2}\left[\left(\nabla^{\dot{A}}\nabla_{\underline{A}} - \nabla_{\underline{A}}\nabla^{\dot{A}}\right)\chi_{\underline{B}\dot{D}}\right]\nabla^{\underline{\dot{D}}}G_{\underline{C\dot{A}}}
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
T_{\underline{A}}{}^{\dot{A}}{}_{\underline{B}\dot{A}}{}^{D}H_{D\underline{C}} = {} & \left[\frac{2}{3}\left(\nabla^2 + \frac{1}{3}R\right)U - \frac{1}{3}\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\overline{U} + \frac{1}{2}\nabla_{E\dot{E}}\chi^{E\dot{E}} + \right. \\
& \left. + \frac{i}{3}\chi^{E\dot{E}}G_{E\dot{E}} + \frac{i}{2}\left(\nabla^{E}\nabla^{\dot{E}} - \nabla^{\dot{E}}\nabla^{E}\right)\chi_{E\dot{E}}\right]W_{ABC}
\end{aligned} \tag{B.5}$$

$$T_{\underline{A}}{}^{\dot{A}}{}_{\underline{B}\dot{A}}{}^{\dot{D}}H_{\dot{D}\underline{C}} = -\frac{i}{12}\left(\nabla_{\underline{A}}G_{\underline{B}}{}^{\dot{D}}\right)\overline{R}\chi_{\underline{C}\dot{D}} \tag{B.6}$$

$$\begin{aligned}
\nabla_{\underline{A}}{}^{\dot{A}}H_{\underline{B}\dot{A}\ \underline{C}} = {} & -\frac{1}{16}\nabla_{\underline{A}}{}^{\dot{A}}\left(\overline{R}\nabla_{\underline{B}}\chi_{\underline{C}\dot{A}}\right) - \frac{1}{24}\nabla_{\underline{A}}{}^{\dot{A}}\left(\chi_{\underline{B}\dot{A}}\nabla_{\underline{C}}\overline{R}\right) + \\
& + \frac{1}{8}\nabla_{\underline{A}}{}^{\dot{A}}\nabla_{\dot{A}}\left(\nabla_{\underline{B}}\nabla^{\dot{E}} - \nabla^{\dot{E}}\nabla_{\underline{B}}\right)\chi_{\underline{C}\dot{E}} + \\
& + \frac{i}{4}\nabla_{\underline{A}}{}^{\dot{A}}\nabla_{\underline{B}}{}^{\dot{E}}\nabla_{\dot{A}}\chi_{\underline{C}\dot{E}} + \frac{3}{16}\nabla_{\underline{A}}{}^{\dot{A}}\left(G_{\underline{B}}{}^{\dot{E}}\nabla_{\dot{E}}\chi_{\underline{C}\dot{A}}\right) + \\
& + \frac{1}{16}\nabla_{\underline{A}}{}^{\dot{A}}\left(G_{\underline{B}\dot{A}}\nabla_{\dot{E}}\chi_{\underline{C}}{}^{\dot{E}}\right) - \frac{5}{16}\nabla_{\underline{A}}{}^{\dot{A}}\left(G_{\underline{B}}{}^{\dot{E}}\nabla_{\dot{A}}\chi_{\underline{C}\dot{E}}\right).
\end{aligned} \tag{B.7}$$

We may then write (B.2) as

$$\begin{aligned}
\delta W_{ABC} = {} & -\frac{1}{2}\left[\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\overline{U}\right]W_{ABC} - \frac{i}{2}\chi^{E\dot{E}}G_{E\dot{E}}W_{ABC} - \\
& - \frac{i}{8}\left[\left(\nabla^{E}\nabla^{\dot{E}} - \nabla^{\dot{E}}\nabla^{E}\right)\chi_{E\dot{E}}\right]W_{ABC} - \\
& - \frac{i}{4}\left(\nabla_{\dot{A}}\nabla_{\underline{A}} - \nabla_{\underline{A}}\nabla_{\dot{A}}\right)\chi^{D\dot{A}}W_{D\underline{BC}} +
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\left(\nabla_{E\dot{E}}\chi^{E\dot{E}}\right)W_{ABC}-\frac{i}{4}\left[\left(\nabla^{\dot{A}}\nabla_{\underline{A}}-\nabla_{\underline{A}}\nabla^{\dot{A}}\right)\chi_{\underline{B}\dot{D}}\right]\nabla^{\dot{D}}G_{\underline{C}\dot{A}}+\\
&+\frac{1}{16}\nabla_{\underline{A}}{}^{\dot{A}}\left(\overline{R}\nabla_{\underline{B}}\chi_{\underline{C}\dot{A}}\right)-\frac{1}{8}\nabla_{\underline{A}}{}^{\dot{A}}\nabla_{\dot{A}}\left(\nabla_{\underline{B}}\nabla^{\dot{E}}-\nabla^{\dot{E}}\nabla_{\underline{B}}\right)\chi_{\underline{C}\dot{E}}-\\
&-\frac{i}{4}\nabla_{\underline{A}}{}^{\dot{A}}\nabla_{\underline{B}}{}^{\dot{E}}\nabla_{\dot{A}}\chi_{\underline{C}\dot{E}}+\frac{1}{24}\nabla_{\underline{A}}{}^{\dot{A}}\left(\chi_{\underline{B}\dot{A}}\nabla_{\underline{C}}\overline{R}\right)-\\
&-\frac{3}{16}\nabla_{\underline{A}}{}^{\dot{A}}\left(G_{\underline{B}}{}^{\dot{E}}\nabla_{\dot{E}}\chi_{\underline{C}\dot{A}}\right)-\frac{1}{16}\nabla_{\underline{A}}{}^{\dot{A}}\left(G_{\underline{B}\dot{A}}\nabla_{\dot{E}}\chi_{\underline{C}}{}^{\dot{E}}\right)+\\
&+\frac{5}{16}\nabla_{\underline{A}}{}^{\dot{A}}\left(G_{\underline{B}}{}^{\dot{E}}\nabla_{\dot{A}}\chi_{\underline{C}\dot{E}}\right)-\frac{i}{24}\left(\nabla_{\underline{A}}G_{\underline{B}}{}^{\dot{D}}\right)\overline{R}\chi_{\underline{C}\dot{D}}+\\
&+\frac{3}{32}i\overline{R}\left(\nabla_{\underline{A}}\chi_{\underline{B}}{}^{\dot{A}}\right)G_{\underline{C}\dot{A}}+\frac{i}{16}\chi_{\underline{B}}{}^{\dot{A}}\left(\nabla_{\underline{A}}\overline{R}\right)G_{\underline{C}\dot{A}}+\\
&+\frac{3}{16}i\left(\nabla^{\dot{A}}\left(\nabla^{\dot{E}}\nabla_{\underline{A}}-\nabla_{\underline{A}}\nabla^{\dot{E}}\right)\chi_{\underline{B}\dot{E}}\right)G_{\underline{C}\dot{A}}+\\
&+\frac{3}{8}\left(\nabla_{\underline{A}}{}^{\dot{E}}\nabla^{\dot{A}}\chi_{\underline{B}\dot{E}}\right)G_{\underline{C}\dot{A}}-\frac{3}{16}iG_{\underline{A}\dot{E}}G_{\underline{C}\dot{A}}\nabla^{\dot{E}}\chi_{\underline{B}}{}^{\dot{A}}
\end{aligned}
\tag{B.8}
$$

and hence

$$
\begin{aligned}
2\overline{W}^2W^{ABC}\delta W_{ABC}={}&-i\chi^{E\dot{E}}G_{E\dot{E}}W^2\overline{W}^2+\frac{1}{2}\left(\nabla_{E\dot{E}}\chi^{E\dot{E}}\right)W^2\overline{W}^2-\\
&-\frac{i}{2}\left[\left(\nabla^{\dot{A}}\nabla_{A}-\nabla_{A}\nabla^{\dot{A}}\right)\chi_{B\dot{D}}\right]\left(\nabla^{\dot{D}}G_{C\dot{A}}\right)W^{ABC}\overline{W}^2-\\
&-\frac{1}{8}\nabla_{A}{}^{\dot{A}}\left(\overline{R}\nabla_{B}\chi_{C\dot{A}}\right)W^{ABC}\overline{W}^2+\\
&+\frac{1}{4}\left[\nabla_{A}{}^{\dot{A}}\nabla_{\dot{A}}\left(\nabla_{B}\nabla^{\dot{E}}-\nabla^{\dot{E}}\nabla_{B}\right)\chi_{C\dot{E}}\right]W^{ABC}\overline{W}^2+\\
&+\frac{i}{2}\left[\nabla_{A}{}^{\dot{A}}\nabla_{B}{}^{\dot{E}}\nabla_{\dot{A}}\chi_{C\ \dot{E}}\right]W^{ABC}\overline{W}^2-\\
&-\frac{1}{12}\left[\nabla_{A}{}^{\dot{A}}\left(\chi_{B\dot{A}}\nabla_{C}\overline{R}\right)\right]W^{ABC}\overline{W}^2+\\
&+\frac{3}{8}\left[\nabla_{A}{}^{\dot{A}}\left(G_{B}{}^{\dot{E}}\nabla_{\dot{E}}\chi_{C\dot{A}}\right)\right]W^{ABC}\overline{W}^2+\\
&+\frac{1}{8}\nabla_{A}{}^{\dot{A}}\left(G_{B\dot{A}}\nabla_{\dot{E}}\chi_{C}{}^{\dot{E}}\right)W^{ABC}\overline{W}^2-\\
&-\frac{5}{8}\nabla_{A}{}^{\dot{A}}\left(G_{B}{}^{\dot{E}}\nabla_{\dot{A}}\chi_{C\ \dot{E}}\right)W^{ABC}\overline{W}^2+\\
&+\frac{i}{12}\left(\nabla_{A}G_{B}{}^{\dot{D}}\right)\overline{R}\chi_{C\dot{D}}W^{ABC}\overline{W}^2-\\
&-\frac{3}{32}i\overline{R}\left(\nabla_{A}\chi_{B}{}^{\dot{A}}\right)G_{C\dot{A}}W^{ABC}\overline{W}^2-\\
&-\frac{3}{8}i\left(\nabla^{\dot{A}}\left(\nabla^{\dot{E}}\nabla_{A}-\nabla_{A}\nabla^{\dot{E}}\right)\chi_{B\dot{E}}\right)G_{C\dot{A}}W^{ABC}\overline{W}^2-\\
&-\frac{3}{4}\left(\nabla_{A}{}^{\dot{E}}\nabla^{\dot{A}}\chi_{B\dot{E}}\right)G_{C\dot{A}}W^{ABC}\overline{W}^2-\\
&-\frac{i}{8}\chi_{B}{}^{\dot{A}}\left(\nabla_{A}\overline{R}\right)G_{C\dot{A}}W^{ABC}\overline{W}^2+
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3}{8} i G_{A\dot{E}} G_{C\dot{A}} \nabla^{\dot{E}} \chi_B{}^{\dot{A}} W^{ABC} \overline{W}^2 - \\
&-\left[\left(\overline{\nabla}^2+\frac{1}{3}\overline{R}\right)\overline{U}\right] W^2 \overline{W}^2 . \qquad \text{(B.9)}
\end{aligned}
$$

C. Exact computations for the simplification of $G_{A\dot{A}}$

Here we present the exact expansion of all the terms of (4.17).

$$
\begin{aligned}
&\left(\nabla_{\dot{A}}\nabla^B-\nabla^B\nabla_{\dot{A}}\right)\nabla^{\dot{D}}\left(G^C{}_{\dot{D}} W_{ABC}\overline{W}^2\right)= \\
&\quad=-\frac{i}{6}\overline{W}^2\left(\nabla^B W_{ABC}\right)\nabla^C{}_{\dot{A}}\overline{R}+\frac{i}{12}\overline{W}^2\nabla^B W_{ABC}\nabla^B\nabla^C{}_{\dot{A}}\overline{R}- \\
&\quad-\frac{5}{24}\overline{W}^2 W_{ABC} G^C{}_{\dot{A}}\nabla^B\overline{R}+\frac{1}{24}\left(\nabla_{\dot{A}}\overline{W}^2\right)\left(\nabla^B W_{ABC}\right)\nabla^C\overline{R}- \\
&\quad-\frac{i}{12}\overline{W}^2\left(\nabla^C{}_{\dot{A}} W_{ABC}\right)\nabla^B\overline{R}-2i\left(\nabla^{\dot{D}}\overline{W}^2\right)W_{ABC}\left(\nabla^B{}_{\underline{\dot{A}}} G^C{}_{\underline{\dot{D}}}\right)- \\
&\quad-2\left(\nabla^{\dot{D}}\overline{W}^2\right)W_{ABC}\left(\nabla^B\nabla_{\underline{\dot{A}}} G^C{}_{\underline{\dot{D}}}\right)-9G^B{}_{\underline{\dot{A}}} G^C{}_{\underline{\dot{D}}}\left(\nabla^{\dot{D}}\overline{W}^2\right)W_{ABC}+ \\
&\quad+4i\left(\nabla^{B\dot{D}}\overline{W}^2\right)W_{ABC}\nabla_{\underline{\dot{A}}} G^C{}_{\underline{\dot{D}}}+2\left(\nabla^{\dot{D}}\overline{W}^2\right)\left(\nabla^B W_{ABC}\right)\nabla_{\underline{\dot{A}}} G^C{}_{\underline{\dot{D}}}- \\
&\quad-\left(\overline{\nabla}^2\overline{W}^2\right)W_{ABC}\nabla^B G^C{}_{\dot{A}}+\left(\overline{\nabla}^2\overline{W}^2\right)\left(\nabla^B W_{ABC}\right)G^C{}_{\dot{A}}+ \\
&\quad+3iG^C{}_{\dot{A}}\left(\nabla^{\dot{D}}\nabla^C{}_{\dot{D}}\overline{W}^2\right)W_{ABC}+2iG^{C\dot{D}}\left(\nabla_{\underline{\dot{D}}}\overline{W}^2\right)\nabla^C{}_{\underline{\dot{A}}} W_{ABC}+ \\
&\quad+2iG^{C\dot{D}}\left(\nabla_{\underline{\dot{D}}}\nabla^C{}_{\underline{\dot{A}}}\overline{W}^2\right)W_{ABC}+\frac{1}{2}\left(\nabla_{\dot{A}}\overline{W}^2\right)W_A{}^{BC}\nabla^D W_{DBC}- \\
&\quad-iG^C{}_{\dot{A}}\left(\nabla^{\dot{D}}\overline{W}^2\right)\nabla^C{}_{\dot{D}} W_{ABC} \qquad \text{(C.1)}
\end{aligned}
$$

$$
\begin{aligned}
&\left(\nabla_{\dot{D}}\nabla^B-\nabla^B\nabla_{\dot{D}}\right)\left(\left(\nabla_{\dot{A}} G^{C\dot{D}}\right)W_{ABC}\overline{W}^2\right)= \\
&\quad=-\frac{i}{6}\overline{W}^2\left(\nabla^B\nabla^C{}_{\dot{A}}\overline{R}\right)W_{ABC}-\frac{i}{6}\overline{W}^2\left(\nabla^C{}_{\dot{A}}\overline{R}\right)\nabla^B W_{ABC}- \\
&\quad-\frac{15}{16}\overline{W}^2\left(\nabla^B\overline{R}\right)G^C{}_{\dot{A}} W_{ABC}+\frac{1}{4}\left(\nabla_{\dot{A}}\overline{W}^2\right)\left(\nabla^B W_{ABC}\right)\nabla^C\overline{R}- \\
&\quad-\frac{i}{24}\overline{W}^2\left(\nabla^B\overline{R}\right)\nabla^C{}_{\dot{A}} W_{ABC}+\frac{i}{24}\left(\nabla^B\overline{R}\right)\left(\nabla^C{}_{\dot{A}}\overline{W}^2\right)W_{ABC}- \\
&\quad-\overline{R}\overline{W}^2 G^C{}_{\dot{A}}\nabla^B W_{ABC}-\frac{5}{6}\overline{R}\overline{W}^2\left(\nabla^B G^C{}_{\dot{A}}\right)W_{ABC}-\frac{1}{2}W^2\overline{W}^2 G_{A\dot{A}}- \\
&\quad-3G^{B\dot{D}}\left(\nabla_{\underline{\dot{A}}} G^C{}_{\underline{\dot{D}}}\right)W_{ABC}\overline{W}^2+2i\overline{W}^2\left(\nabla_{\underline{\dot{A}}} G^C{}_{\underline{\dot{D}}}\right)\nabla^{B\dot{D}} W_{ABC}- \\
&\quad-2i\left(\nabla_{\underline{\dot{A}}} G^C{}_{\underline{\dot{D}}}\right)\left(\nabla^{B\dot{D}}\overline{W}^2\right)W_{ABC}-2\left(\nabla_{\underline{\dot{A}}} G^C{}_{\underline{\dot{D}}}\right)\left(\nabla^{\dot{D}}\overline{W}^2\right)\nabla^B W_{ABC}- \\
&\quad-2\left(\nabla^B\nabla_{\underline{\dot{A}}} G^C{}_{\underline{\dot{D}}}\right)\left(\nabla^{\dot{D}}\overline{W}^2\right)W_{ABC}+2i\overline{W}^2 W^{BCD}\nabla_{D\dot{A}} W_{ABC} \qquad \text{(C.2)}
\end{aligned}
$$

$$
\begin{aligned}
&\nabla^{\dot{D}}\left(G^B{}_{\dot{A}} G^C{}_{\dot{D}} W_{ABC}\overline{W}^2\right)= \\
&\quad=-\frac{1}{16}\overline{W}^2\left(\nabla^B\overline{R}\right)G^C{}_{\dot{A}} W_{ABC}-G^{B\dot{D}}\left(\nabla_{\underline{\dot{A}}} G^C{}_{\underline{\dot{D}}}\right)W_{ABC}\overline{W}^2+
\end{aligned}
$$

$$+G^B{}_{\underline{\dot A}}G^C{}_{\underline{\dot D}}\left(\nabla^{\dot D}\overline{W}^2\right)W_{ABC} \tag{C.3}$$

$$\begin{aligned}
&\nabla^{\dot D}\nabla^B{}_{\dot A}\left(G^C{}_{\dot D}W_{ABC}\overline{W}^2\right) = \\
&\quad = -\frac{1}{24}\overline{W}^2\left(\nabla^B\nabla^C{}_{\dot A}\overline{R}\right)W_{ABC} + \frac{11}{96}i\overline{W}^2\left(\nabla^B\overline{R}\right)G^C{}_{\dot A}W_{ABC} - \\
&\qquad - \frac{1}{24}\overline{W}^2\left(\nabla^B\overline{R}\right)\nabla^C{}_{\dot A}W_{ABC} - \frac{1}{24}\left(\nabla^B\overline{R}\right)\left(\nabla^C{}_{\dot A}\overline{W}^2\right)W_{ABC} - \\
&\qquad - \frac{i}{12}\overline{R}\left(\nabla^B G^C{}_{\dot A}\right)\overline{W}^2W_{ABC} - \frac{i}{12}\overline{R}G^C{}_{\dot A}\overline{W}^2\nabla^B W_{ABC} + \\
&\qquad + \frac{3}{2}iG^{B\dot D}\left(\nabla_{\underline{\dot A}}G^C{}_{\underline{\dot D}}\right)W_{ABC}\overline{W}^2 - G^{B\dot D}\left(\nabla_{\underline{\dot A}}\nabla^C{}_{\underline{\dot D}}\overline{W}^2\right)W_{ABC} - \\
&\qquad - G^{B\dot D}\left(\nabla_{\underline{\dot A}}\overline{W}^2\right)\nabla^C{}_{\underline{\dot D}}W_{ABC} + \frac{1}{2}G^B{}_{\dot A}\left(\nabla^{\dot D}\overline{W}^2\right)\nabla^C{}_{\dot D}W_{ABC} + \\
&\qquad + \frac{1}{2}G^B{}_{\dot A}\left(\nabla^{\dot D}\nabla^C{}_{\dot D}\overline{W}^2\right)W_{ABC} - iG_{A\dot A}W^2\overline{W}^2 + \\
&\qquad + \left(\nabla^B{}_{\underline{\dot A}}G^C{}_{\underline{\dot D}}\right)\left(\nabla^{\dot D}\overline{W}^2\right)W_{ABC} + \frac{i}{4}\left(\nabla_{\dot A}\overline{W}^2\right)W_A{}^{BC}\nabla^D W_{DBC}
\end{aligned} \tag{C.4}$$

$$\begin{aligned}
&\nabla^B\left(\overline{R}\nabla^C{}_{\dot A}\left(W_{ABC}\overline{W}^2\right)\right) = \\
&\quad = \overline{W}^2\left(\nabla^B\overline{R}\right)\nabla^C{}_{\dot A}W_{ABC} + \left(\nabla^B\overline{R}\right)\left(\nabla^C{}_{\dot A}\overline{W}^2\right)W_{ABC} + \\
&\qquad + \overline{R}\overline{W}^2\nabla^B\nabla^C{}_{\dot A}W_{ABC} + \overline{R}\left(\nabla^C{}_{\dot A}\overline{W}^2\right)\nabla^B W_{ABC}
\end{aligned} \tag{C.5}$$

$$\begin{aligned}
&\nabla^{\dot D}\left(G^C{}_{\dot D}\nabla^B{}_{\dot A}\left(W_{ABC}\overline{W}^2\right)\right) = \\
&\quad = -\frac{1}{24}\left(\nabla^B\overline{R}\right)\left(\nabla^C{}_{\dot A}\overline{W}^2\right)W_{ABC} - \frac{1}{24}\overline{W}^2\left(\nabla^B\overline{R}\right)\nabla^C{}_{\dot A}W_{ABC} - \\
&\qquad - G^{B\dot D}\left(\nabla_{\underline{\dot A}}\nabla^C{}_{\underline{\dot D}}\overline{W}^2\right)W_{ABC} + \frac{1}{2}G^B{}_{\dot A}\left(\nabla^{\dot D}\nabla^C{}_{\dot D}\overline{W}^2\right)W_{ABC} - \\
&\qquad - G^{B\dot D}\left(\nabla_{\underline{\dot A}}\overline{W}^2\right)\nabla^C{}_{\underline{\dot D}}W_{ABC} + \frac{1}{2}G^B{}_{\dot A}\left(\nabla^{\dot D}\overline{W}^2\right)\nabla^C{}_{\dot D}W_{ABC} - \\
&\qquad - \frac{i}{12}\overline{R}G^C{}_{\dot A}\overline{W}^2\nabla^B W_{ABC} - \frac{5}{96}iG^C{}_{\dot A}\overline{W}^2\left(\nabla^B\overline{R}\right)W_{ABC} - \\
&\qquad - \frac{5}{2}iG^{B\dot D}\left(\nabla_{\underline{\dot A}}G^C{}_{\underline{\dot D}}\right)W_{ABC}\overline{W}^2
\end{aligned} \tag{C.6}$$

$$\begin{aligned}
&\nabla_{\dot A}\left(G^C{}_{\dot D}\nabla^{B\dot D}\left(W_{ABC}\overline{W}^2\right)\right) = \\
&\quad = \left(\nabla_{\underline{\dot A}}G^C{}_{\underline{\dot D}}\right)W_{ABC}\nabla^{B\dot D}\overline{W}^2 + \overline{W}^2\left(\nabla_{\underline{\dot A}}G^C{}_{\underline{\dot D}}\right)\nabla^{B\dot D}W_{ABC}
\end{aligned} \tag{C.7}$$

$$\begin{aligned}
&\nabla^{\dot D}\left(G^B{}_{\dot A}\nabla^C{}_{\dot D}\left(W_{ABC}\overline{W}^2\right)\right) = \\
&\quad = -\frac{1}{48}\left(\nabla^B\overline{R}\right)\left(\nabla^C{}_{\dot A}\overline{W}^2\right)W_{ABC} - \frac{1}{48}\left(\nabla^B\overline{R}\right)\overline{W}^2\nabla^C{}_{\dot A}W_{ABC} - \\
&\qquad - \frac{5}{48}i\left(\nabla^B\overline{R}\right)G^C{}_{\dot A}\overline{W}^2W_{ABC} - \frac{i}{6}\overline{R}G^C{}_{\dot A}\overline{W}^2\nabla^B W_{ABC} - \\
&\qquad - \left(\nabla_{\underline{\dot A}}G^C{}_{\underline{\dot D}}\right)W_{ABC}\nabla^{B\dot D}\overline{W}^2 - \overline{W}^2\left(\nabla_{\underline{\dot A}}G^C{}_{\underline{\dot D}}\right)\nabla^{B\dot D}W_{ABC} +
\end{aligned}$$

$$+ G^{B}{}_{\dot A}\left(\nabla^{\dot D}\overline{W}^{2}\right)\nabla^{C}{}_{\dot D}W_{ABC} + G^{B}{}_{\dot A}\left(\nabla^{\dot D}\nabla^{C}{}_{\dot D}\overline{W}^{2}\right)W_{ABC} \tag{C.8}$$

$$\begin{aligned}
&\nabla^{\dot D}\nabla^{B}{}_{\dot A}\nabla^{C}{}_{\dot D}\left(W_{ABC}\overline{W}^{2}\right) = \\
&\quad = -\frac{5}{48}i\overline{W}^{2}\left(\nabla^{B}\nabla^{C}{}_{\dot A}\overline{R}\right)W_{ABC} - \frac{i}{6}\overline{W}^{2}\left(\nabla^{C}{}_{\dot A}\overline{R}\right)\nabla^{B}W_{ABC} - \\
&\qquad - \frac{5}{32}\overline{W}^{2}G^{C}{}_{\dot A}\left(\nabla^{B}\overline{R}\right)W_{ABC} - \frac{5}{32}i\left(\nabla^{B}\overline{R}\right)\left(\nabla^{C}{}_{\dot A}\overline{W}^{2}\right)W_{ABC} - \\
&\qquad - \frac{5}{96}i\left(\nabla^{B}\overline{R}\right)\overline{W}^{2}\nabla^{C}{}_{\dot A}W_{ABC} - \frac{i}{4}\overline{RW}^{2}\nabla^{B}\nabla^{C}{}_{\dot A}W_{ABC} - \\
&\qquad - \frac{i}{4}\overline{R}\left(\nabla^{C}{}_{\dot A}\overline{W}^{2}\right)\nabla^{B}W_{ABC} - \frac{1}{4}\overline{RW}^{2}G^{C}{}_{\dot A}\nabla^{B}W_{ABC} - \\
&\qquad - \frac{5}{12}\overline{RW}^{2}\left(\nabla^{B}G^{C}{}_{\dot A}\right)W_{ABC} - \frac{1}{2}\left(\nabla^{B}\nabla_{\underline{\dot A}}G^{C}{}_{\underline{\dot D}}\right)\left(\nabla^{\dot D}\overline{W}^{2}\right)W_{ABC} - \\
&\qquad - i\left(\nabla^{B}{}_{\underline{\dot A}}G^{C}{}_{\underline{\dot D}}\right)\left(\nabla^{\dot D}\overline{W}^{2}\right)W_{ABC} - \frac{5}{2}\overline{W}^{2}G^{B\dot D}\left(\nabla_{\underline{\dot A}}G^{C}{}_{\underline{\dot D}}\right)W_{ABC} - \\
&\qquad - \frac{5}{2}i\left(\nabla^{B\dot D}\overline{W}^{2}\right)\left(\nabla_{\underline{\dot A}}G^{C}{}_{\underline{\dot D}}\right)W_{ABC} + \frac{i}{2}\overline{W}^{2}\left(\nabla_{\underline{\dot A}}G^{C}{}_{\underline{\dot D}}\right)\nabla^{B\dot D}W_{ABC} + \\
&\qquad + \frac{1}{4}\left(\overline{\nabla}^{2}\overline{W}^{2}\right)\left(\nabla^{B}G^{C}{}_{\dot A}\right)W_{ABC} + \left(\nabla^{\dot D}\nabla^{B}{}_{\underline{\dot A}}\nabla^{C}{}_{\underline{\dot D}}\overline{W}^{2}\right)W_{ABC} + \\
&\qquad + \left(\nabla^{\dot D}\overline{W}^{2}\right)\nabla^{B}{}_{\underline{\dot A}}\nabla^{C}{}_{\underline{\dot D}}W_{ABC} + \frac{3}{2}\left(\nabla^{\dot D}\nabla^{C}{}_{\dot D}\overline{W}^{2}\right)\nabla^{B}{}_{\dot A}W_{ABC} - \\
&\qquad - \left(\nabla_{\underline{\dot A}}\nabla^{C}_{\underline{\dot D}}\overline{W}^{2}\right)\nabla^{B\dot D}W_{ABC} - 2i\overline{W}^{2}W_{A}{}^{BD}\nabla^{C}{}_{\dot A}W_{ABC} + \\
&\qquad + \frac{1}{8}\left(\nabla_{\dot A}\overline{W}^{2}\right)W_{A}{}^{BC}\nabla^{D}W_{DBC} + \frac{1}{48}\left(\nabla_{\dot A}\overline{W}^{2}\right)\left(\nabla^{B}\overline{R}\right)\nabla^{C}W_{ABC}
\end{aligned} \tag{C.9}$$

$$\begin{aligned}
&\left(\nabla_{\dot A}\nabla^{B} - \nabla^{B}\nabla_{\dot A}\right)\nabla^{\dot D}\nabla^{C}{}_{\dot D}\left(W_{ABC}\overline{W}^{2}\right) = \\
&\quad = \frac{5}{24}\overline{W}^{2}\left(\nabla^{B}\nabla^{C}{}_{\dot A}\overline{R}\right)W_{ABC} + \frac{1}{12}\overline{W}^{2}\left(\nabla^{C}{}_{\dot A}\overline{R}\right)\nabla^{B}W_{ABC} - \\
&\qquad - \frac{55}{48}i\overline{W}^{2}G^{C}{}_{\dot A}\left(\nabla^{B}\overline{R}\right)W_{ABC} + \frac{3}{8}\left(\nabla^{B}\overline{R}\right)\left(\nabla^{C}{}_{\dot A}\overline{W}^{2}\right)W_{ABC} + \\
&\qquad + \frac{11}{24}\left(\nabla^{B}\overline{R}\right)\overline{W}^{2}\nabla^{C}{}_{\dot A}W_{ABC} - \frac{5}{48}i\left(\nabla_{\dot A}\overline{W}^{2}\right)\left(\nabla^{C}\overline{R}\right)\nabla^{B}W_{ABC} + \\
&\qquad + \frac{1}{2}\overline{R}\left(\nabla^{C}{}_{\dot A}\overline{W}^{2}\right)\nabla^{B}W_{ABC} - \frac{5}{6}i\overline{RW}^{2}\left(\nabla^{B}G^{C}{}_{\dot A}\right)W_{ABC} - \\
&\qquad - \frac{5}{6}i\overline{RW}^{2}G^{C}{}_{\dot A}\nabla^{B}W_{ABC} + \frac{1}{3}\overline{RW}^{2}\nabla^{B}\nabla^{C}{}_{\dot A}W_{ABC} - \\
&\qquad - 2i\left(\nabla^{B}\nabla_{\underline{\dot A}}G^{C}{}_{\underline{\dot D}}\right)\left(\nabla^{\dot D}\overline{W}^{2}\right)W_{ABC} \\
&\qquad - 6\left(\nabla^{B}{}_{\underline{\dot A}}G^{C}{}_{\underline{\dot D}}\right)\left(\nabla^{\dot D}\overline{W}^{2}\right)W_{ABC} + \\
&\qquad + 8\left(\nabla^{B\dot D}\overline{W}^{2}\right)\left(\nabla_{\underline{\dot A}}G^{C}{}_{\underline{\dot D}}\right)W_{ABC} - \frac{3}{2}i\left(\overline{\nabla}^{2}\overline{W}^{2}\right)\left(\nabla^{B}G^{C}{}_{\dot A}\right)W_{ABC} + \\
&\qquad + 5i\left(\nabla^{\dot D}\overline{W}^{2}\right)\left(\nabla_{\underline{\dot A}}G^{C}{}_{\underline{\dot D}}\right)\nabla^{B}W_{ABC} + 6G^{B\dot D}\left(\nabla_{\underline{\dot A}}\overline{W}^{2}\right)\nabla^{C}{}_{\underline{\dot D}}W_{ABC} - \\
&\qquad - 6G^{B}{}_{\dot A}\left(\nabla^{\dot D}\overline{W}^{2}\right)\nabla^{C}{}_{\dot D}W_{ABC} + 10G^{B\dot D}\left(\nabla_{\underline{\dot A}}\nabla^{C}{}_{\underline{\dot D}}\overline{W}^{2}\right)W_{ABC} -
\end{aligned}$$

$$\begin{aligned}
&-4G^{B}{}_{\dot A}\left(\nabla^{\dot D}\nabla^{C}{}_{\dot D}\overline{W}^2\right)W_{ABC}+\left(\overline{\nabla}^2\nabla^{B}{}_{\dot A}\overline{W}^2\right)\nabla^{C}W_{ABC}+\\
&+i\left(\nabla^{\dot D}\nabla^{C}{}_{\dot D}\overline{W}^2\right)\nabla^{B}{}_{\dot A}W_{ABC}-\frac{5}{4}i\left(\nabla_{\dot A}\overline{W}^2\right)W_{A}{}^{BC}\nabla^{D}W_{DBC}+\\
&+2i\left(\nabla_{\underline{\dot A}}\nabla^{C}{}_{\underline{\dot D}}\overline{W}^2\right)\nabla^{B\dot D}W_{ABC}+\left(\overline{\nabla}^2\overline{W}^2\right)\nabla^{B}\nabla^{C}{}_{\dot A}W_{ABC}-\\
&-2W^2\nabla_{A\dot A}\overline{W}^2-2i\left(\nabla^{\dot D}\overline{W}^2\right)\nabla^{B}{}_{\underline{\dot A}}\nabla^{C}{}_{\underline{\dot D}}W_{ABC}+\\
&+2i\left(\nabla^{\dot D}\nabla^{B}{}_{\underline{\dot A}}\nabla^{C}{}_{\underline{\dot D}}\overline{W}^2\right)W_{ABC}\,. \qquad\qquad (\mathrm{C}.10)
\end{aligned}$$

D. Exact computations to first order in α

Here we present the expansion, to first order in α, of all the terms of (4.27):

$$\nabla^{C}{}_{\dot A}\overline{R}=6\alpha\left(\nabla^{C}{}_{\dot A}W^2\right)\overline{\nabla}^2\overline{W}^2+6\alpha W^2\overline{\nabla}^2\nabla^{C}{}_{\dot A}\overline{W}^2+\mathcal{O}\left(\alpha^2\right) \qquad (\mathrm{D}.1)$$

$$\begin{aligned}
\nabla^{\underline{B}}\nabla^{\underline{C}}{}_{\dot A}\overline{R}={}&6\alpha\left(\nabla^{\underline{B}}\nabla^{\underline{C}}{}_{\dot A}W^2\right)\overline{\nabla}^2\overline{W}^2+24i\alpha\left(\nabla^{\underline{C}}{}_{\dot A}W^2\right)\nabla^{\dot C}\nabla^{\underline{B}}{}_{\dot C}\overline{W}^2+\\
&+6\alpha\left(\nabla^{\underline{B}}W^2\right)\overline{\nabla}^2\nabla^{\underline{C}}{}_{\dot A}\overline{W}^2+24i\alpha W^2\nabla^{\dot C}\nabla^{\underline{B}}{}_{\dot C}\nabla^{\underline{C}}{}_{\dot A}\overline{W}^2-\\
&-48\alpha W^2W^{BCD}\nabla_{D\dot A}\overline{W}^2+\mathcal{O}\left(\alpha^2\right) \qquad (\mathrm{D}.2)
\end{aligned}$$

$$\begin{aligned}
\nabla^{\underline{B}}G^{\underline{C}}{}_{\dot A}={}&i\alpha\overline{W}^2\nabla^{\underline{B}}\nabla^{\underline{C}}{}_{\dot A}W^2+i\alpha\left(\nabla^{\underline{B}}\overline{W}^2\right)\nabla^{\underline{C}}{}_{\dot A}W^2+\\
&+\frac{3}{2}i\alpha\left(\nabla^{E\dot E}\nabla_{E\dot E}\nabla_{D\dot A}\overline{W}^2\right)W^{BCD}\\
&+3i\alpha W^{F}{}_{DE}\left(\nabla_{F\dot A}\overline{W}^2\right)\nabla^{\underline{B}}W^{\underline{CDE}}\\
&+3i\alpha W_{\dot A}{}^{\dot E\dot F}\left(\nabla_{\dot E}\nabla_{D\dot F}\overline{W}^2\right)W^{BCD}+\frac{3}{4}\alpha W_{\dot A\dot B\dot C}\nabla^2\nabla^{\underline{B}\dot B}\nabla^{\underline{C}\dot C}W^2+\\
&+\frac{3}{2}\alpha\left(\nabla^{\dot C}\nabla_{D\dot C}\nabla_{E\dot A}\overline{W}^2\right)\nabla^{\underline{B}}W^{\underline{CDE}}+\mathcal{O}\left(\alpha^2\right) \qquad (\mathrm{D}.3)
\end{aligned}$$

$$\begin{aligned}
\nabla^{\underline{B}}\nabla_{\underline{\dot A}}G^{\underline{C}}{}_{\underline{\dot B}}={}&-i\alpha\left(\nabla^{\underline{B}}W^2\right)\nabla_{\underline{\dot A}}\nabla^{\underline{C}}{}_{\underline{\dot B}}\overline{W}^2-2\alpha W^2\nabla^{\underline{B}}{}_{\underline{\dot A}}\nabla^{\underline{C}}{}_{\underline{\dot B}}\overline{W}^2-\\
&-2\alpha\left(\nabla^{\underline{B}}{}_{\underline{\dot A}}W^2\right)\nabla^{\underline{C}}{}_{\underline{\dot B}}\overline{W}^2+i\alpha\left(\nabla_{\underline{\dot A}}\overline{W}^2\right)\nabla^{\underline{B}}\nabla^{\underline{C}}{}_{\underline{\dot B}}W^2+\\
&+\frac{3}{2}i\alpha W_{\dot A\dot B\dot C}\nabla^{\underline{B}}\nabla^{E\dot E}\nabla_{E\dot E}\nabla^{\underline{C}\dot C}W^2-\\
&-\frac{3}{4}\alpha\left(\nabla^2\nabla^{\underline{B}\dot D}\nabla^{\underline{C}\dot C}W^2\right)\nabla_{\underline{\dot A}}W_{\underline{\dot B\dot C\dot D}}+\\
&+3i\alpha\left(\nabla^{D}\nabla_{D}{}^{\dot D}\nabla^{\underline{C}\dot C}W^2\right)\nabla^{\underline{B}}{}_{\underline{\dot A}}W_{\underline{\dot B\dot C\dot D}}+\\
&+\frac{3}{4}\alpha\left(\overline{\nabla}^2\nabla_{D\underline{\dot B}}\nabla_{E\underline{\dot A}}\overline{W}^2\right)\nabla^{\underline{B}}W^{\underline{CDE}}-\\
&-3i\alpha W^{\underline{C}DE}\nabla^{\dot C}\nabla^{\underline{B}}{}_{\dot C}\nabla_{D\underline{\dot B}}\nabla_{E\underline{\dot A}}\overline{W}^2-\\
&-\frac{3}{2}\alpha W^{BCD}W^{\dot D}{}_{\dot A\dot B}\overline{\nabla}^2\nabla_{D\dot D}\overline{W}^2-\\
&-12\alpha W^{\underline{\underline{B}}DE}W^{\underline{\underline{C}}}{}_{EF}\nabla_{\underline{D\dot A}}\nabla^{\underline{F}}{}_{\underline{\dot B}}\overline{W}^2+
\end{aligned}$$

$$\begin{aligned}
&+3i\alpha W^{BCD}\left(\nabla^{\dot{C}}\nabla_{D}{}^{\dot{D}}\overline{W}^2\right)\nabla_{\underline{\dot{A}}}W_{\underline{\dot{B}\dot{C}\dot{D}}}-\\
&-3i\alpha W_{\dot{A}\dot{B}\dot{C}}\left(\nabla_{D}\nabla_{E}{}^{\dot{C}}W^2\right)\nabla^{\underline{B}}W^{\underline{CDE}}+\\
&+\frac{3}{2}i\alpha W_{\dot{A}\dot{B}\dot{C}}W^{BCD}\nabla^2\nabla_{D}{}^{\dot{C}}W^2-\\
&-6\alpha\left(\nabla^{\underline{B}}{}_{\underline{\dot{A}}}W_{\underline{\dot{B}\dot{C}\dot{D}}}\right)W^{\dot{C}\dot{D}\dot{E}}\nabla^{\underline{C}}{}_{\dot{E}}W^2+\\
&+3i\alpha\left(\nabla_{\underline{\dot{A}}}W_{\underline{\dot{B}\dot{C}\dot{D}}}\right)W^{\dot{C}\dot{D}\dot{E}}\nabla^{\underline{B}}\nabla^{\underline{C}}{}_{\dot{E}}W^2+\mathcal{O}\left(\alpha^2\right)
\end{aligned}\tag{D.4}$$

$$\begin{aligned}
\nabla^{\underline{B}}{}_{\underline{\dot{A}}}G^{\underline{C}}{}_{\underline{\dot{B}}}={}&-\alpha\left(\nabla_{\underline{\dot{A}}}\overline{W}^2\right)\nabla^{\underline{B}}\nabla^{\underline{C}}{}_{\underline{\dot{B}}}W^2-i\alpha W^2\nabla^{\underline{B}}{}_{\underline{\dot{A}}}\nabla^{\underline{C}}{}_{\underline{\dot{B}}}\overline{W}^2-\\
&-\frac{3}{2}\alpha\left(\nabla^{\dot{C}}\nabla_{D\dot{C}}\nabla_{E\underline{\dot{B}}}\overline{W}^2\right)\nabla^{\underline{B}}{}_{\underline{\dot{A}}}W^{\underline{CDE}}+\\
&+\frac{3}{2}\alpha W^{\underline{C}EF}\nabla^{\dot{C}}\nabla^{\underline{B}}{}_{\dot{C}}\nabla_{E\underline{\dot{A}}}\nabla_{F\underline{\dot{B}}}\overline{W}^2+\\
&+\frac{3}{4}\alpha W^{BCD}\nabla_{\underline{\dot{A}}}\nabla^{E\dot{E}}\nabla_{E\dot{E}}\nabla_{D\underline{\dot{B}}}\overline{W}^2+\\
&+\frac{3}{2}\alpha\left(\nabla_{\underline{\dot{A}}}\nabla^{F}{}_{\underline{\dot{B}}}\overline{W}^2\right)W_{DEF}\nabla^{\underline{B}}W^{\underline{CDE}}+\\
&+3i\alpha\left(\nabla^{F}{}_{\underline{\dot{B}}}\overline{W}^2\right)W_{DEF}\nabla^{\underline{B}}{}_{\underline{\dot{A}}}W^{\underline{CDE}}-\\
&-6i\alpha W^{\underline{\underline{B}}DE}W^{\underline{\underline{C}}}{}_{EF}\nabla_{\underline{D\dot{A}}}\nabla^{\underline{F}}{}_{\underline{\dot{B}}}\overline{W}^2+\text{h.c.}+\mathcal{O}\left(\alpha^2\right).
\end{aligned}\tag{D.5}$$

References

[1] M.B. Green and J.H. Schwarz, *Anomaly cancellation in supersymmetric $D = 10$ gauge theory and superstring theory*, *Phys. Lett.* **B 149** (1984) 117.

[2] M.B. Green and J.H. Schwarz, *Infinity cancellations in* SO(32) *superstring theory*, *Phys. Lett.* **B 151** (1985) 21; *The hexagon gauge anomaly in type-I superstring theory*, *Nucl. Phys.* **B 255** (1985) 93.

[3] D.J. Gross and E. Witten, *Superstring modifications of Einstein's equations*, *Nucl. Phys.* **B 277** (1986) 1.

[4] D.J. Gross and J.H. Sloan, *The quartic effective action for the heterotic string*, *Nucl. Phys.* **B 291** (1987) 41.

[5] M.T. Grisaru, A.E.M. van de Ven and D. Zanon, *Four loop beta function for the $N = 1$ and $N = 2$ supersymmetric nonlinear sigma model in two-dimensions*, *Phys. Lett.* **B 173** (1986) 423; *Two-dimensional supersymmetric sigma models on Ricci flat Kähler manifolds are not finite*, *Nucl. Phys.* **B 277** (1986) 388; *Four loop divergences for the $N = 1$ supersymmetric nonlinear sigma model in two-dimensions*, *Nucl. Phys.* **B 277** (1986) 409;
M.T. Grisaru and D. Zanon, *Sigma model superstring corrections to the Einstein-Hilbert action*, *Phys. Lett.* **B 177** (1986) 347.

[6] J.G. Russo and A.A. Tseytlin, *One-loop four-graviton amplitude in eleven-dimensional supergravity*, *Nucl. Phys.* **B 508** (1997) 245 [`hep-th/9707134`].

[7] M.B. Green, M. Gutperle and P. Vanhove, *One loop in eleven dimensions*, *Phys. Lett.* **B 409** (1997) 177 [`hep-th/9706175`].

[8] M.B. Green, H.-h. Kwon and P. Vanhove, *Two loops in eleven dimensions*, *Phys. Rev.* **D 61** (2000) 104010 [`hep-th/9910055`].

[9] E. Bergshoeff, A. Salam and E. Sezgin, *Supersymmetric R^2 actions, conformal invariance and Lorentz Chern-Simons term in six-dimensions and ten-dimensions*, *Nucl. Phys.* **B 279** (1987) 659.

[10] E. Bergshoeff and M. de Roo, *Supersymmetric Chern-Simons terms in ten-dimensions*, *Phys. Lett.* **B 218** (1989) 210; *The quartic effective action of the heterotic string and supersymmetry*, *Nucl. Phys.* **B 328** (1989) 439.

[11] L. Bonora, P. Pasti and M. Tonin, *Superspace formulation of 10d SUGRA+SYM theory a la Green-Schwarz*, *Phys. Lett.* **B 188** (1987) 335;
L. Bonora, M. Bregola, K. Lechner, P. Pasti and M. Tonin, *Anomaly free supergravity and super Yang-Mills theories in ten-dimensions*, *Nucl. Phys.* **B 296** (1988) 877.

[12] M. Raciti, F. Riva and D. Zanon, *Perturbative approach to $D = 10$ superspace supergravity with a Lorentz Chern-Simons form*, *Phys. Lett.* **B 227** (1989) 118.

[13] P. Fré and I. Pesando, *Supersymmetrization of the Lorentz Chern-Simons term in $D = 10$*, `SISSA 65/91/EP` in *Strings and symmetries*, Stony Brook, 1991, N. Berkovits et al. eds., World Scientific, 1992.

[14] M. de Roo, H. Suelmann and A. Wiedemann, *The supersymmetric effective action of the heterotic string in ten-dimensions*, *Nucl. Phys.* **B 405** (1993) 326 [`hep-th/9210099`].

[15] K. Peeters, P. Vanhove and A. Westerberg, *Supersymmetric higher-derivative actions in ten and eleven dimensions, the associated superalgebras and their formulation in superspace*, *Class. and Quant. Grav.* **18** (2001) 843 [`hep-th/0010167`]; *Supersymmetric R^4 actions and quantum corrections to superspace torsion constraints*, `hep-th/0010182`.

[16] M. Cederwall, U. Gran, M. Nielsen and B.E.W. Nilsson, *Manifestly supersymmetric M-theory*, *J. High Energy Phys.* **10** (2000) 041 [`hep-th/0007035`];
Generalised 11-dimensional supergravity, `hep-th/0010042`.

[17] S. Cecotti, S. Ferrara, L. Girardello and M. Porrati, *Lorentz Chern-Simons terms in $N = 1$ four-dimensional supergravity consistent with supersymmetry and string compactification*, *Phys. Lett.* **B 164** (1985) 46;
S. Cecotti, S. Ferrara, L. Girardello, M. Porrati and A. Pasquinucci, *Matter coupling in higher derivative supergravity*, *Phys. Rev.* **D 33** (1986) 2504.

[18] S. Cecotti, S. Ferrara and M. Villasante, *Linear multiplets and super Chern-Simons forms in 4-d supergravity*, *Int. J. Mod. Phys.* **A 2** (1987) 1839.

[19] R. D'Auria, P. Fré, G. de Matteis and I. Pesando, *Superspace constraints and Chern-Simons cohomology in $D = 4$ superstring effective theories*, *Int. J. Mod. Phys.* **A 4** (1989) 3577.

[20] R. Grimm, J. Wess and B. Zumino, *A complete solution of the Bianchi identities in superspace*, *Nucl. Phys.* **B 152** (1979) 255.

[21] J. Wess and B. Zumino, *Superfield lagrangian for supergravity*, *Phys. Lett.* **B 74** (1978) 51.

[22] S.J. Gates Jr., and W. Siegel, *Understanding constraints in superspace formulations of supergravity*, *Nucl. Phys.* **B 163** (1980) 519.

[23] B. Zumino, in *Recent developments in gravitation*, Cargèse, 1978, M. Lévy and S. Deser eds., Plenum Press, 1979.

[24] P. van Nieuwenhuizen and P. West, *Principles of supersymmetry and supergravity*, forthcoming book to be published by Cambridge University Press.

[25] P. Binétruy, G. Girardi and R. Grimm,y *Supergravity couplings: a geometric formulation*, *Phys. Rep.* **343** (2001) 255 [`hep-th/0005225`].

[26] S.J. Gates Jr., M.T. Grisaru, M. Roček and W. Siegel, *Superspace or one thousand and one lessons in supersymmetry*, Benjamin/Cummings 1983, `hep-th/0108200`.

[27] R. Wald, *General relativity*, Chicago University Press 1984, chapters 4 and 13.

[28] F. Moura, work in progress.

[29] K.S. Stelle and P.C. West, *Minimal auxiliary fields for supergravity*, *Phys. Lett.* **B 74** (1978) 330.

[30] S. Ferrara and P. van Nieuwenhuizen, *The auxiliary fields of supergravity*, *Phys. Lett.* **B 74** (1978) 333.

[31] J. Wess and B. Zumino, *The component formalism follows from the superspace formulation of supergravity*, *Phys. Lett.* **B 79** (1978) 394.

[32] S.J. Gates Jr. and S.V. Ketov, *4D, $N = 1$ Born-Infeld supergravity*, *Class. and Quant. Grav.* **18** (2001) 3561 [`hep-th/0104223`].

[33] B.E.W. Nilsson, *Simple ten-dimensional supergravity in superspace*, *Nucl. Phys.* **B 188** (1981) 176.

[34] B.E.W. Nilsson and A.K. Tollsten, *Superspace formulation of the ten-dimensional coupled Einstein Yang-Mills system*, *Phys. Lett.* **B 171** (1986) 212.

[35] G.F. Chapline and N.S. Manton, *Unification of Yang-Mills theory and supergravity in ten-dimensions*, *Phys. Lett.* **B 120** (1983) 105.

[36] E. Cremmer and S. Ferrara, *Formulation of eleven-dimensional supergravity in superspace*, *Phys. Lett.* **B 91** (1980) 61;
L. Brink and P. Howe, *Eleven-dimensional supergravity on the mass-shell in superspace*, *Phys. Lett.* **B 91** (1980) 384.

[37] S.J. Gates Jr. and H. Nishino, *Deliberations on 11-d superspace for the M-theory effective action*, *Phys. Lett.* **B 508** (2001) 155 [`hep-th/0101037`].

[38] H. Nishino and S. Rajpoot, *A note on embedding M-theory corrections into eleven-dimensional superspace*, `hep-th/0103224`.